Mechanics of Materials

Chapter 2

ANALYSIS OF STRESS:
CONCEPTS & DEFINITIONS

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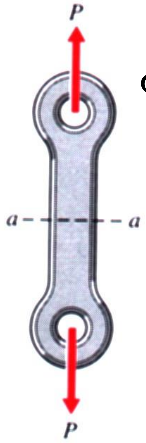
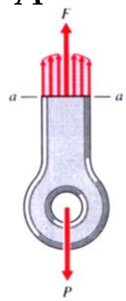
2-2 Normal Stress Under Axial Loading

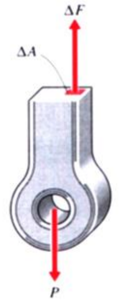
Definition: Stress = $\dfrac{\text{Force}}{\text{Area}}$

intensity of force



$$\sigma_{avg} = \frac{F}{A}$$



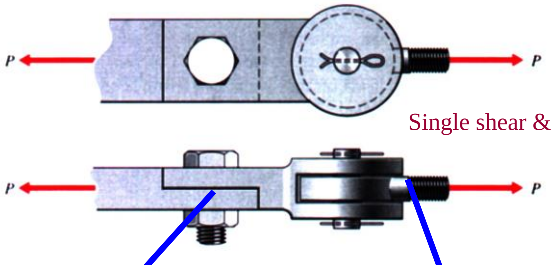


$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

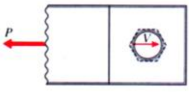
σ: "+": tensile stress
"–": compressive stress

2-3 Shearing Stress in Connections

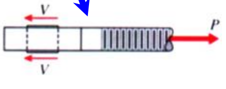
- Connections: rivets, bolts, pins, nails, welds



Single shear &



$$\tau_{avg} = \frac{V}{A}$$



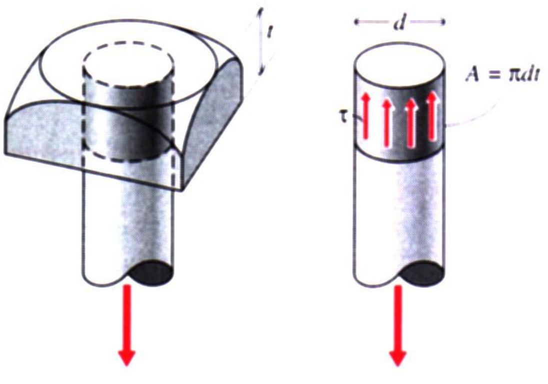
$$\tau = \lim_{\Delta A \to 0} \frac{\Delta V}{\Delta A}$$

(Note: The shear stress τ **cannot** be uniformly distributed over *A*.)

2-3 Shearing Stress in Connections

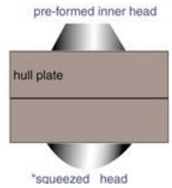
- Punching Shear

periphery area = πdt



(a) (b)

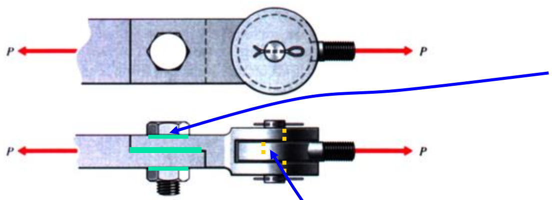
http://thcarmor.theshoppe.com/images/rivet.jpg

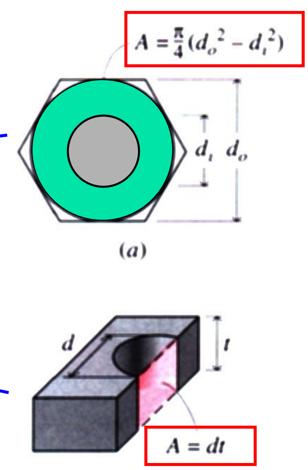


www.mrs.org/microworld/ titanic.html

2-4 Bearing Stress

- Bearing stress: **compressive normal stress**





$$\sigma_b = \frac{F}{A}$$

2-5 Units of Stress

- USCS: United State Customary System (British units)
- SI: International System of Units
- Unit of **stress**: force per unit area (**intensity** of force)
  - psi: **p**ound per **s**quare **i**nch
  - ksi: **k**ilo pound per **s**quare **i**nch
    - pound (lb_f): 1 slug · 1 ft/s^2
    - kip: **ki**lo **p**ound
  - Pa: Pascal, Newton per square meter
    - N: Newton, 1 kg · 1 m/s^2 = 0.2248 lb_f
  - MPa: mega (10^6) Newton per square meter
    (1 bar = 0.1 MPa)

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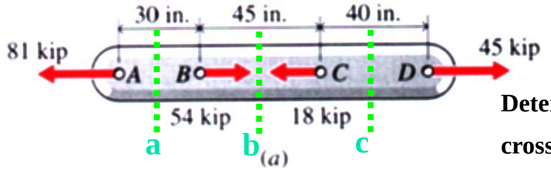
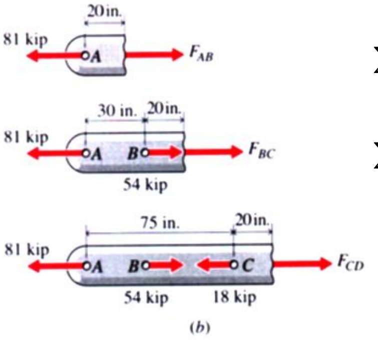
Example Problem 2-1(a)



$A = 3.00\ in^2$

Determine normal stress at cross-section a, b, and c=?



$\sum F = F_{AB} - 81 = 0$ $\quad F_{AB} = 81\ \text{kip (T)}$

$\sum F = F_{BC} + 54 - 81 = 0$ $\quad F_{BC} = 27\ \text{kip (T)}$

$F_{CD} = ?$

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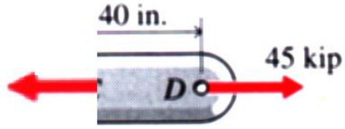
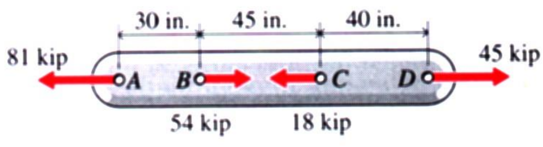
Which one is correct?

- 1. F_{CD} = - 45 kip
- 2. F_{CD} = 45 kip
- 3. F_{CD} = - 18 kip
- 4. F_{CD} = 18 kip
- 5. No correct answer

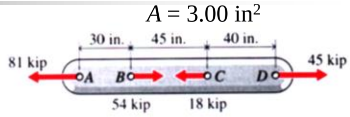




(a)

Example Problem 2-1(b)

$A = 3.00 \text{ in}^2$



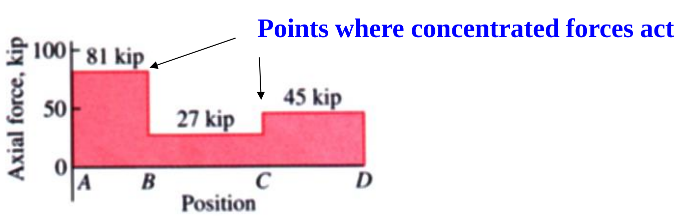
(a)

$\sigma_{CD} = ?$

1. **- 45 ksi**
2. **+ 45 ksi**
3. **- 15 ksi**
4. **+ 15 ksi**
5. **No correct answer**

$$\sigma_{CD} = \frac{F_{CD}}{A} = \frac{+45}{3.00} = +15.00 \text{ ksi} = 15.00 \text{ ksi (T)}$$

- Can you draw the axial-force Diagram?



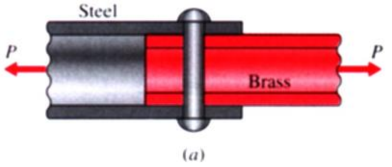
(c)

Example Problem 2-3 (a)

P = 10 kips

$d_s^i = 2.00\ \text{in}$

$t_s = 0.250\ \text{in}$



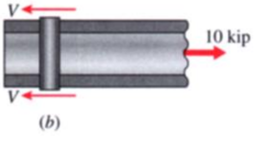
(a)

$d_b^o = 2.00\ \text{in}$

$t_b = 0.275\ \text{in}$

$d_p = 0.750\ \text{in}$

- Determine shearing stress in the pin = ?

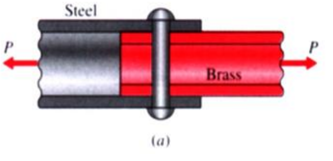


(b)

$2V = 10 \qquad V = 5\ \text{kips}$

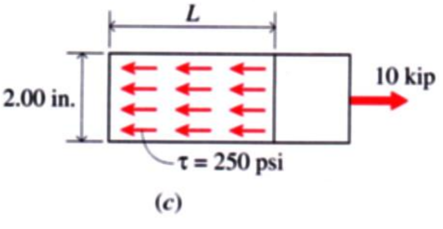
$$\tau = \frac{V}{A} = \frac{5}{\frac{\pi}{4}(0.750)^2} = 11.317\ \text{ksi} \cong 11.32\ \text{ksi}$$

Example Problem 2-3 (b)



(a)

- Replace the pin by a glued joint.
  Length of glued joint = ?



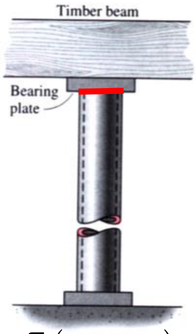
(c)

$A = \pi d L = \pi(2.00)L = 2.00\pi L\ \text{in}^2$

$$\tau = \frac{V}{A} = \frac{10{,}000}{2.00\pi L} = 250\ \text{psi}$$

$L = 6.366\ \text{in} \cong 6.37\ \text{in}$

Example Problem 2-4 (a)



$d_s^o = 150\ \text{mm}$

$t_s = 15\ \text{mm}$

axial load = 150 kN

- Determine average bearing stress between column and bearing plate = ?

$$A = \frac{\pi}{4}\left(d_o^2 - d_i^2\right) = \frac{\pi}{4}\left(150^2 - 120^2\right) = 6362\ \text{mm}^2 = 6362(10^{-6})\ \text{m}^2$$

$$\sigma_b = \frac{F}{A} = \frac{150(10^3)}{6362(10^{-6})} = 23.58(10^6)\ \text{N/m}^2 \cong 23.6\ \text{MPa}$$

Example Problem 2-4 (b)

- Assuming average bearing stress of bearing plate $\leq 3.25\text{MPa}$

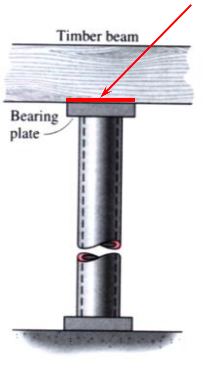
Determine diameter of bearing plate ***d*** = ?

$$A = \frac{\pi}{4} d^2$$

$$\sigma_b = \frac{F}{A} = \frac{150(10^3)}{(\pi/4)d^2} = 3.25(10^6)$$

$$d = 242.4(10^{-3})\ \text{m} \cong 242\ \text{mm}$$

d = ?



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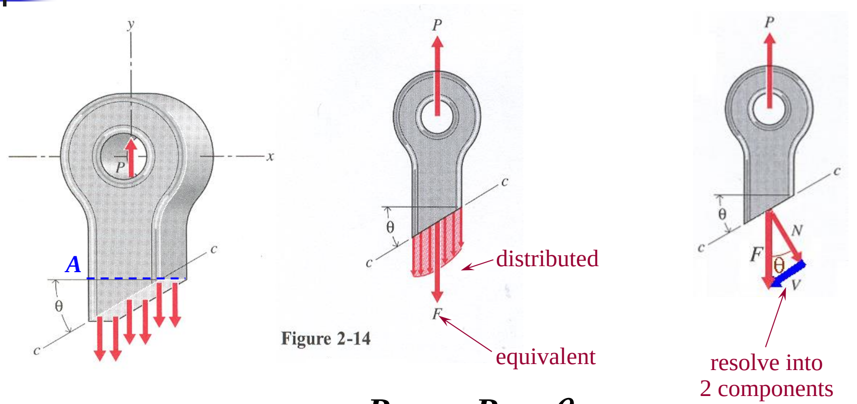
2-6 Stresses on an *Inclined Plane* in an Axially Loaded Member



Average total stress $\sigma_{avg} = \dfrac{P}{A/\cos\theta} = \dfrac{P\cos\theta}{A}$

Not useful for design!

$|N| = P\cos\theta$

$|V| = P\sin\theta$

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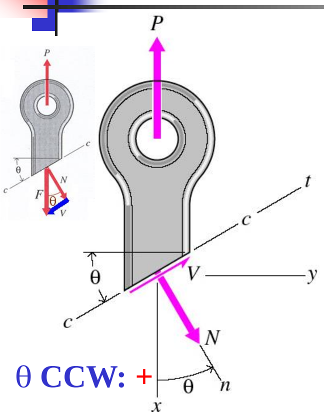
2-6 Stresses on an Inclined Plane in an Axially Loaded Member



θ CCW: +

$N = P\cos\theta$

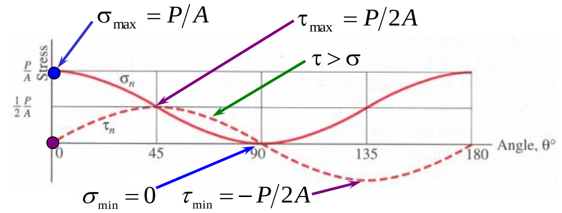
$V = \mathbf{-}\,P\sin\theta$

Axially loaded only!

Assumption: stress uniformly distributed

$$\sigma_n = \frac{N}{A_n} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta = \frac{P}{2A}(1+\cos 2\theta) \qquad (2\text{-}7)$$

$$\tau_n = \frac{V}{A_n} = -\frac{P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta = -\frac{P}{2A}\sin 2\theta \qquad (2\text{-}8)$$



σ and τ vanish at 90°

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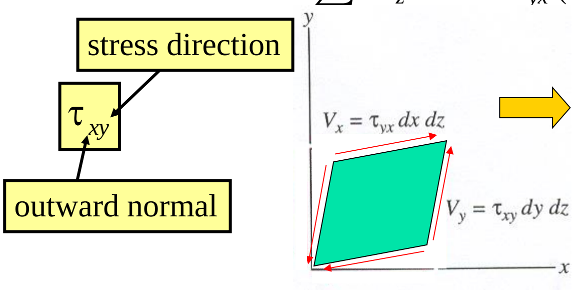
Remarks for τ_{xy} & τ_{yx}

- The equality of τ_n on orthogonal planes can be obtained as follows: If only shear stresses are acting on the surfaces,

$$\sum M_z = 0: \ \tau_{yx}(dx\,dz)dy = \tau_{xy}(dy\,dz)dx$$



$$\tau_{yx} = \tau_{xy}$$

Shear stresses tend to "skew" the element diagonally.

- $\tau_{yx} = \tau_{xy}$ is true even when there are normal stresses.

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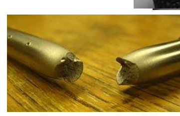
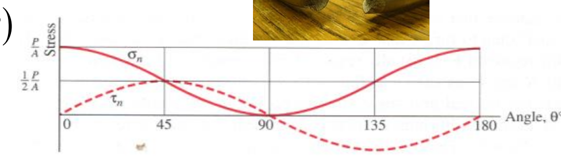
Remarks for τ_{xy} & τ_{yx}

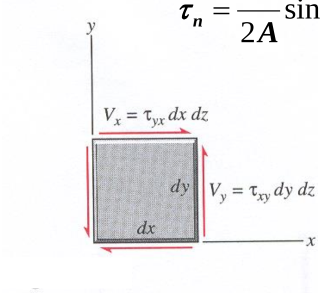
- Failure plane under axial tension loading:
  - brittle material: $\theta \sim 0°$
  - ductile material: $\theta \sim 45°$

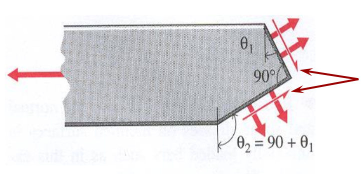


- $\tau(\theta) = -\tau(\theta + 90°)$

$$\tau_n = \frac{-P}{2A}\sin 2\theta$$







Equal magnitude
Change sense (sign)

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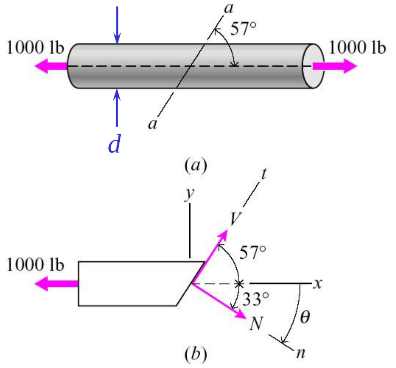
Example Problem 2-6

Given: d = 1.25 in.



- Find: σ_n, τ_n

$$\sigma_n = \frac{N}{A_n} = \frac{P}{2A}(1+\cos 2\theta) \qquad \text{(2-7)}$$

$$= \frac{1000}{2\cdot\pi\cdot 1.25^2/4}\{1+\cos[2\cdot(-33°)]\}$$

$$= 573 \text{ psi}$$

$$\tau_{nt} = \frac{-P}{2A}\sin 2\theta \qquad \text{(2-8)}$$

$$= \frac{-1000}{2\cdot\pi\cdot 1.25^2/4}\sin[2\cdot(-33°)]$$

$$= 372 \text{ psi}$$

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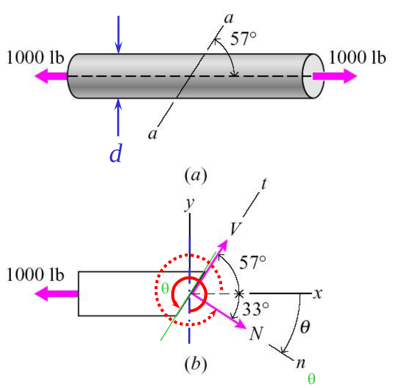
Example Problem 2-6

For d = 1.25 in.



Alternatively,

$$\sigma_n = \frac{P}{2A}(1+\cos 2\theta) \qquad \text{(2-7)}$$

$$= \frac{1000}{2\cdot\pi\cdot 1.25^2/4}[1+\cos(2\cdot 327°)]$$

$$= 573 \text{ psi}$$

$$\tau_{nt} = \frac{-P}{2A}\sin 2\theta \qquad \text{(2-8)}$$

$$= \frac{-1000}{2\cdot\pi\cdot 1.25^2/4}\sin(2\cdot 327°)$$

$$= 372 \text{ psi}$$

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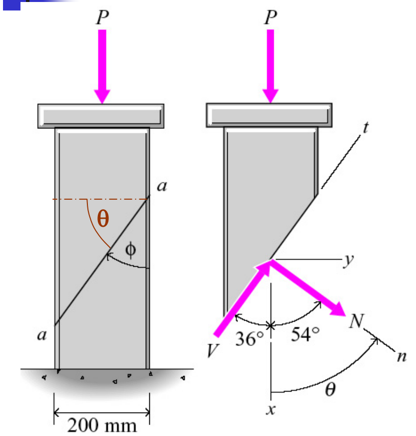
Example Problem 2-7



Given: rectangular block with cross section: 200 × 100 mm,
$\phi = 36^o$,
$\sigma_n = 12.00$ MPa (C) at section *a-a*

- Find:
  - $P = ?$
  - On plane *a-a*, $\tau_n = ?$
  - Maximum normal and shearing stresses in the block = ?

Greek Symbols

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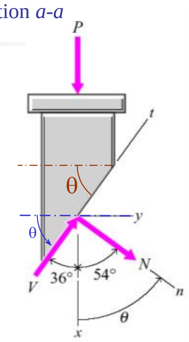
Example Problem 2-7

$\phi = 36^o$,
$\sigma_n = 12.00$ MPa (C) at section *a-a*



$$A = 200(100) = 20{,}000\ \text{mm}^2 = 0.0200\ \text{m}^2$$

$$A_n = A/\cos\theta = 0.020/\cos 54° = 0.03403\ \text{m}^2$$

$$N = \sigma_n A_n = 12(10^6)(0.03403) = 408.4\ \text{kN} \quad \text{(C)}$$

$$\sum F_n = -408.4 + P\cos 54^\circ = 0$$

$$P = 694.8\ \text{kN} \cong 695\ \text{kN} \quad \text{(C)}$$

$$\tau_n = \frac{-P}{2A}\sin 2\theta = \frac{694.8(10^3)}{2(0.0200)}\sin 108° = 16.520(10^6)\ N/m^2 = 16.52\ MPa \quad \text{(eq.2-8)}$$

$$\sigma_{\max} = ?$$

$$\tau_{\max} = ?$$

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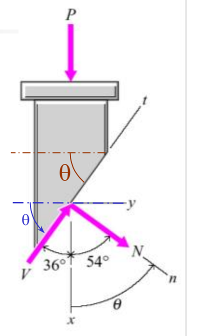
Example Problem 2-7

$\sigma_{\max} = ?\ \tau_{\max} = ?$

1. σ_{max} = 24.3 MPa; τ_{max} = 18.25 Mpa
2. σ_{max} = -24.3 MPa; τ_{max} = 18.25 Mpa
3. σ_{max} = 34.7 MPa; τ_{max} = 17.35 Mpa
4. σ_{max} = -34.7 MPa; τ_{max} = 17.35 Mpa
5. None is correct



$$\sigma_{\max} = \frac{P}{A} = \frac{694.8(10^3)}{0.0200} = 34.74(10^6)\ \text{N/m}^2 \cong 34.7\ \text{MPa (C)}\quad \text{(eq.2-9)}\quad \text{at } \theta = 0°$$

$$\tau_{\max} = \frac{P}{2A} = \frac{694.8(10^3)}{2(0.0200)} = 17.35(10^6)\ \text{N/m}^2 \cong 17.35\ \text{MPa}\quad \text{(eq.2-10)}\quad \text{at } \theta = 45°$$

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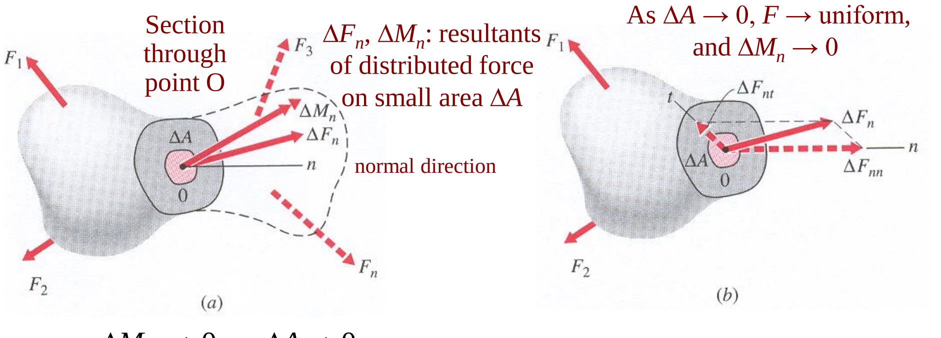
2-7 Stress at a General Point in an Arbitrarily Loaded Member

Stress distribution should not necessarily be uniform in a cross-section.



$\Delta M_n \to 0$ as $\Delta A \to 0$

$$S_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}\quad \text{stress vector}$$

$$\begin{cases} \sigma_n = \lim_{\Delta A \to 0} \dfrac{\Delta F_{nn}}{\Delta A} & \text{normal stress} \\ \tau_n = \lim_{\Delta A \to 0} \dfrac{\Delta F_{nt}}{\Delta A} & \text{shear stress} \end{cases}$$

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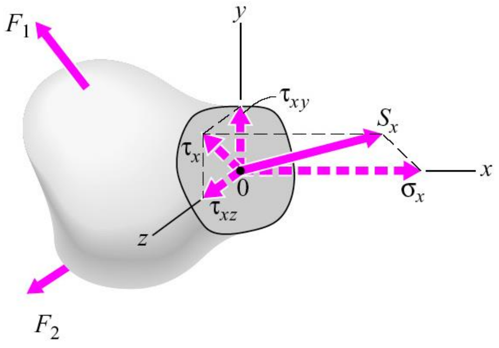
2-7 Stress at a General Point in an Arbitrarily Loaded Member



Resolve τ_x into components τ_{xy} and τ_{xz}

$$S_n = \sigma_n \mathbf{e}_n + \tau_n \mathbf{e}_t$$

$$= \sigma_x \mathbf{e}_x + \tau_{xy} \mathbf{e}_y + \tau_{xz} \mathbf{e}_z$$

Co-plane stress components are vectors.

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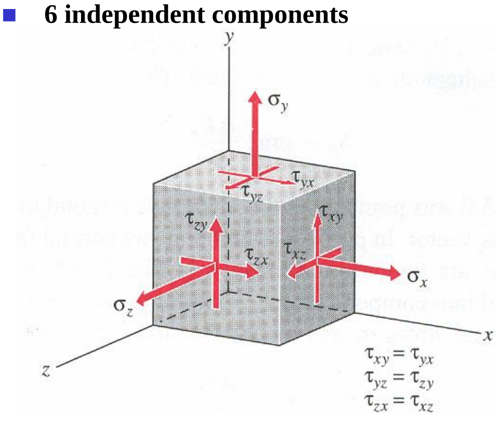
Remarks

- **Resolve S_n on three mutually perpendicular planes**

  ⇒ the state of stress at a point is completely described.
- **Sign convention**
- **6 independent components**



outward normal

τ_{xy}

stress direction

σ_x: normal stress

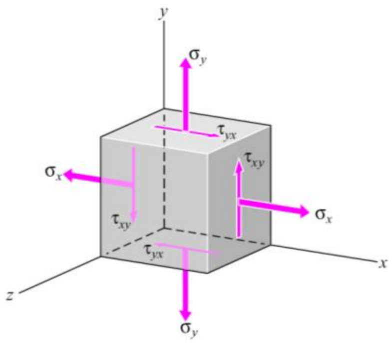
2-8 Two-Dimensional or *Plane Stress*

All forces are confined on a plane, for example, *x-y* plane

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

which implies

$$\tau_{xz} = \tau_{yz} = 0$$

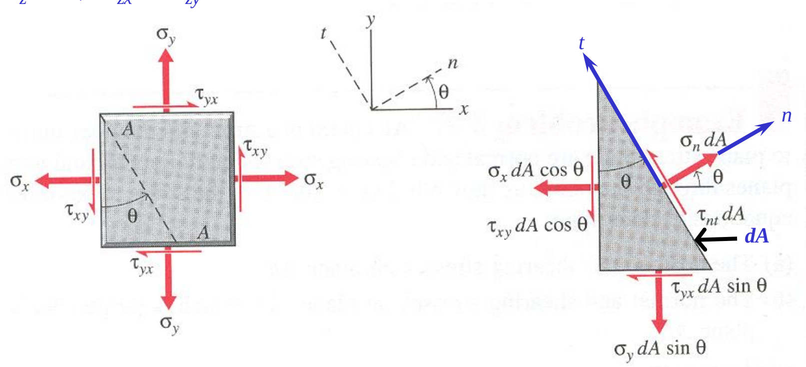


Plane stress occurs at

- points on the outside surface of a body
- points within thin plates where the *z*-components of force are zero.

2-9 The Stress Transformation Equations for Plane Stress

$$\sigma_z = 0,\ \tau_{zx} = \tau_{zy} = 0$$



(*a*) (*b*)

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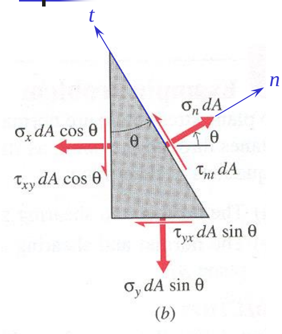
2-9 The Stress Transformation Equations for Plane Stress



$$\overset{+}{\nearrow}\ \sum F_n = \sigma_n dA - \sigma_x(dA\cos\theta)\cos\theta - \sigma_y(dA\sin\theta)\sin\theta$$

$$-\tau_{yx}(dA\sin\theta)\cos\theta - \tau_{xy}(dA\cos\theta)\sin\theta = 0$$

$$\because\ \tau_{yx} = \tau_{xy}$$

$$\boxed{\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta} \quad \text{(2-12a)}$$

$$= \sigma_x\frac{1+\cos 2\theta}{2} + \sigma_y\frac{1-\cos 2\theta}{2} + 2\tau_{xy}\frac{\sin 2\theta}{2}$$

$$= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{(2-12b)}$$

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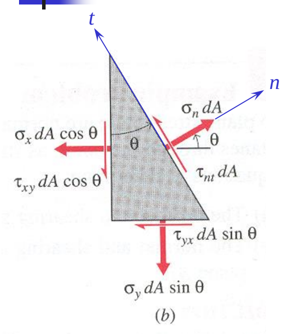
2-9 The Stress Transformation Equations for Plane Stress



$$\overset{+}{\nwarrow}\ \sum F_t = \tau_{nt} dA + \sigma_x(dA\cos\theta)\sin\theta - \sigma_y(dA\sin\theta)\cos\theta$$

$$-\tau_{xy}(dA\cos\theta)\cos\theta + \tau_{yx}(dA\sin\theta)\sin\theta = 0$$

$$\because\ \tau_{yx} = \tau_{xy}$$

$$\boxed{\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)}$$

$$\tau_{nt} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad \text{(2-13b)}$$

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{(2-12b)}$$

For Plane stress

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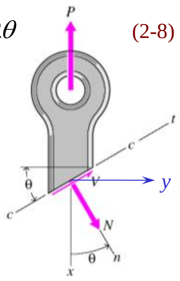
Comparison

2-6 Stresses on an Inclined Plane in an Axially Loaded Member

$$\sigma_n = \frac{P}{2A}\left(1+\cos 2\theta\right) \qquad \text{(2-7)}$$

$$\frac{P}{A} = \sigma_x; \quad \sigma_y = 0; \quad \tau_{xy} = 0$$

$$\tau_n = \frac{-P}{2A}\sin 2\theta \qquad \text{(2-8)}$$



2-8 Plane Stresses

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_n = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos 2\theta$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{nt} = -\frac{\sigma_x}{2}\sin 2\theta$$

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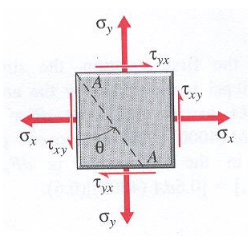
Remarks

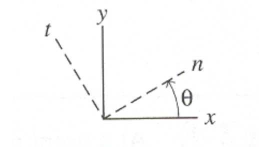
Sign conventions:

- positive stresses τ_{ij} or $\tau_{(-i)(-j)}$, and +θ CCW from +*x*-axis

Positive in the **positive** direction of the j-axis

Positive in the **negative** direction of the j-axis





- (*n*, *t*, *z*) axes have the same order as the (*x*,*y*,*z*) axes. Both are right-hand coordinate system.

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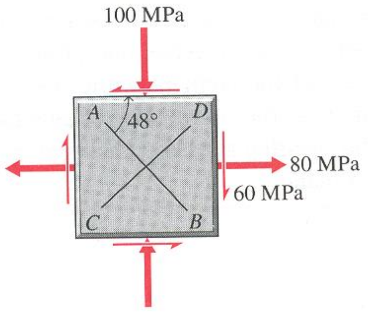
Example Problem 2-8 (I)



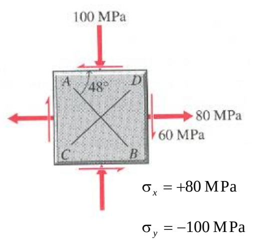
- Given: $\sigma_x = +80\ \text{MPa}$

  $\sigma_y = -100\ \text{MPa}$

  $\tau_{xy} = -60\ \text{MPa}$

- Find: σ_n and τ_{nt} on plane *AB* = ?

  σ_n and τ_{nt} on plane *CD* = ?

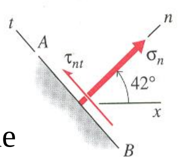


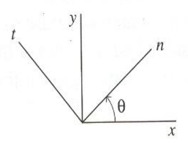
$\sigma_x = +80\ \text{MPa}$

$\sigma_y = -100\ \text{MPa}$

$\tau_{xy} = -60\ \text{MPa}$

Example Problem 2-8 (II)





On plane *AB*

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= 80\cos^2(+42°) + (-100)\sin^2(+42°) + 2(-60)\sin(+42°)\cos(+42°)$$

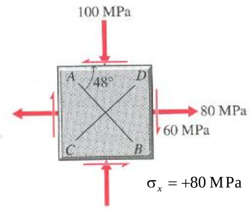
$$= -60.26\ \text{MPa} \cong 60.3\ \text{MPa (C)}$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -[80 - (-100)]\sin(+42°)\cos(+42°) + (-60)(\cos^2(+42°) - \sin^2(+42°))$$

$$= -95.78\ \text{MPa} \cong -95.8\ \text{MPa}$$

Example Problem 2-8 (III)

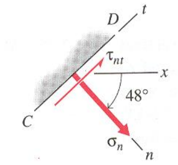


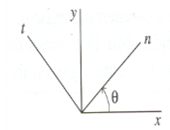
$\sigma_x = +80\ \text{MPa}$

$\sigma_y = -100\ \text{MPa}$

$\tau_{xy} = -60\ \text{MPa}$

On plane *CD*





$\theta = -48°$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= 80\cos^2(-48°) + (-100)\sin^2(-48°) + 2(-60)\sin(-48°)\cos(-48°)$$

$$= +40.26\ \text{MPa} \cong 40.3\ \text{MPa (T)}$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -[80 - (-100)]\sin(-48°)\cos(-48°) + (-60)(\cos^2(-48°) - \sin^2(-48°))$$

$$= +95.78\ \text{MPa} \cong +95.8\ \text{MPa}$$

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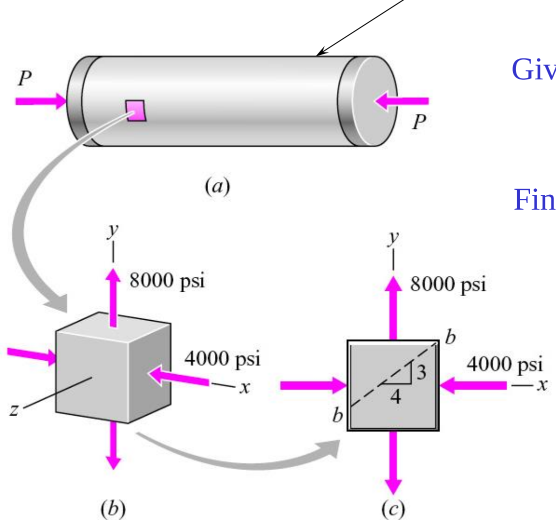
Example Problem 2-9

thin-walled pressure vessel



Given: $\sigma_x = -4000$ psi,
$\sigma_y = 8000$ psi

Find : stresses on plane *b-b*

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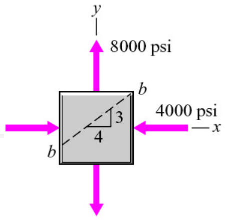
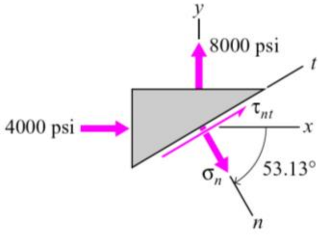


Example Problem 2-9

From Eqs. (2-12a) and (2-13a),



$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

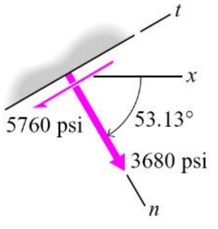
$$= -4000 \cdot \cos^2(-53.13°) + 8000 \cdot \sin^2(-53.13°) + 0$$

$$= 3680 \text{ psi (T)}$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -(-4000 - 800) \cdot \sin(-53.13°) \cdot \cos(-53.13°) + 0$$

$$= -5760 \text{ psi (C)}$$



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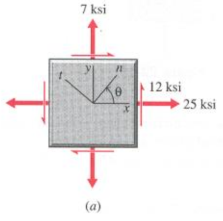
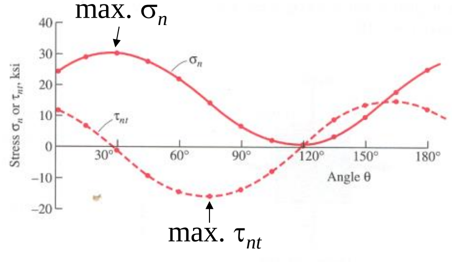
2-10 Principal Stresses and Maximum Shearing Stress – Plane Stress





- Max. σ_n and τ_{nt} (magnitude) can be obtained by plotting curves.
  - ⇨ It's time consuming and inefficient

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Principal Stresses (I)

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

maximize σ_n

$$\Rightarrow \frac{d\sigma_n}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\Rightarrow \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \Rightarrow \quad \theta_{p1}, \ \theta_{p2} \quad (\theta_{p1} = \theta_{p2} + 90^o)$$

principal directions (principal planes)

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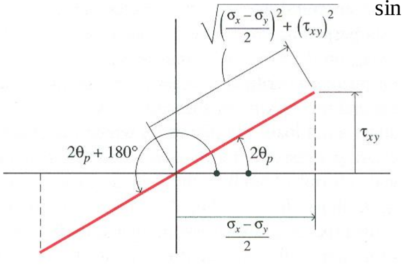
Principal Stresses (II)

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



$$\sin 2\theta_p = \pm\frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \qquad \cos 2\theta_p = \pm\frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

substitute into $\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$

$\Rightarrow$ **$\tau_{nt}(\theta_p) = 0$!!**

(Principal planes: **Planes free of shear stress**)

substitute into $\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

$$\Rightarrow \sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Two principal stresses

$$\sigma_{p1} + \sigma_{p2} = \sigma_x + \sigma_y$$

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Maximum Shearing Stress (I)

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

maximize τ_{nt}

$$\Rightarrow \frac{d\tau_{nt}}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\Rightarrow \tan 2\theta_\tau = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\Rightarrow \theta_{\tau 1},\ \theta_{\tau 2}$$

($\theta_{\tau 1}$ and $\theta_{\tau 2}$ are 90° apart)

$$\tan 2\theta_\tau \times \tan 2\theta_p = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \times \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -1$$

(θ_p and θ_τ are 45° apart)

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Maximum Shearing Stress (II)

$$\sin 2\theta_\tau = \pm\frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}, \qquad \cos 2\theta_\tau = \pm\frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

($\theta_{\tau 1}$ and $\theta_{\tau 2}$ are 90° apart)

$$\tau(\theta) = -\tau(\theta + 90°)$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Two maximum shearing stresses with opposite sign.

$$\left(\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\right)$$

$$\Rightarrow \tau_{max} = (\tau_p) = \frac{\sigma_{p1} - \sigma_{p2}}{2}$$

τ_{max} (or τ_p): maximum in-plane shearing stress

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Remarks (I)

$$\frac{d\sigma_n}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta \quad (= 0 \text{ for } \sigma_{max})$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

- $\frac{d\sigma_n}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 2\tau_{nt}$

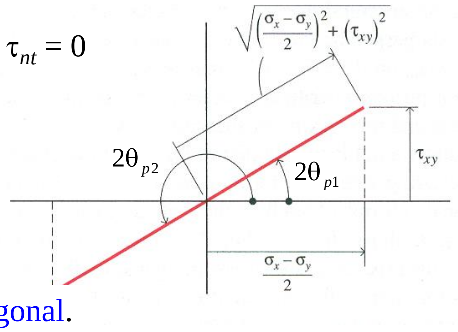
  ⇨ On principal planes, $\tau_{nt} = 0$

  (or $\tau_{nt}(\theta_p) = 0$!!)

- $\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$

  ⇨ $\theta_{p2} = \theta_{p1} + 90°$

  Principal planes are orthogonal.



- For plane stress problems, $\sigma_{p3} = \sigma_z = 0$
- $\sigma_{p1} + \sigma_{p2} = \sigma_x + \sigma_y$ ← $\left(\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \right)$

  The sum of two normal stresses on two orthogonal planes are **invariant**.

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Remarks (II)

- $\tan 2\theta_p > 0$ ⇨ $0° < \theta_p < 45°$  along z–axis

  $\tan 2\theta_p < 0$ ⇨ $-45° < \theta_p < 0°$  along z–axis

- Numerically greater σ_p will act on the plane that makes an angle of 45° or less with the plane of the numerically larger of σ_x and σ_y

- $\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ $\quad$ $\tan 2\theta_\tau = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$ (Reciprocal relationship)

  ⇨ $\theta_\tau = \theta_p + 45°$ ← $\left(\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \right)$

- On θ_τ, $\quad \tau_{max} = \frac{\sigma_{p1} - \sigma_{p2}}{2}$, $\quad \boxed{\sigma_n = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_{p1} + \sigma_{p2}}{2}}$

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Remarks (III)

- 3D case:
  - There are 3 orthogonal planes on which $\tau_{nt} = 0$

    ⇨ principal planes

  - $\tau_{max} = \dfrac{\sigma_{max} - \sigma_{min}}{2}$

    acts on planes that bisect the angles between the planes of σ_{max} and σ_{min} (see next page)

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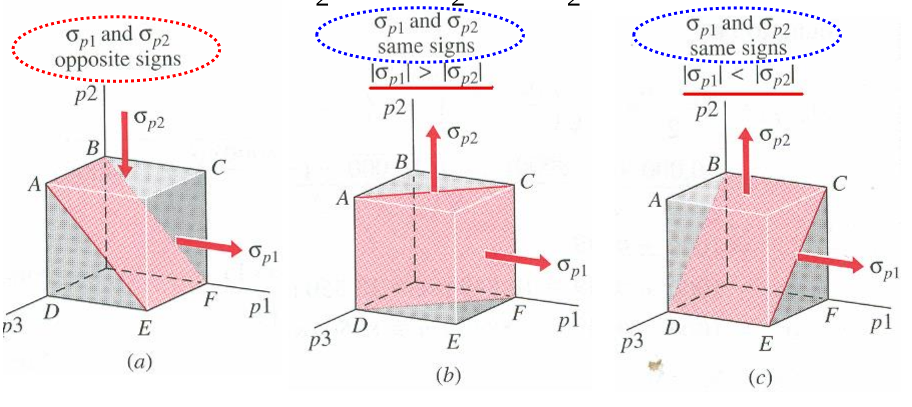
Remarks (IV)

- For **plane stress**:

  max. σ_n : σ_{p1} , σ_{p2} , 0 ← σ_{p3}

  max. τ_{nt} : $\dfrac{\sigma_{p1}-\sigma_{p2}}{2}$, $\dfrac{\sigma_{p1}-0}{2}$, $\dfrac{\sigma_{p2}-0}{2}$



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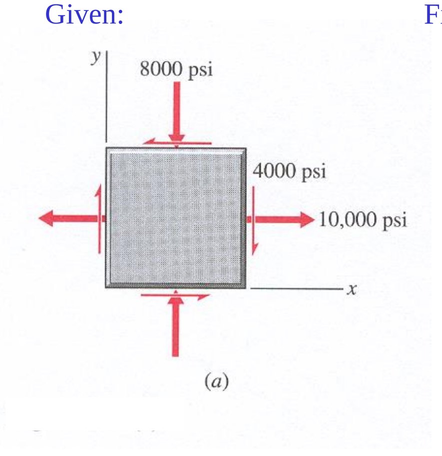
Example Problem 2-11

Given:



(a)

Find:

- σ_p, τ_{max} =?
- θ_p, θ_τ = ?
- Show the stresses on a sketch.

σ_x = + 10,000 psi

σ_y = – 8000 psi

τ_{xy} = – 4000 psi

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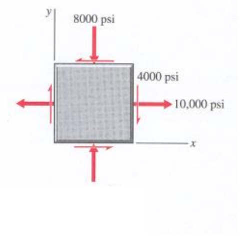
Example Problem 2-11



$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{(2-15)}$$

$$= \frac{10000 + (-8000)}{2} \pm \sqrt{\left(\frac{10000 - (-8000)}{2}\right)^2 + (-4000)^2}$$

$$= 1000 \pm 9849$$

$$\sigma_{p1} = 1000 + 9849 = 10{,}849 \text{ psi} \cong 10{,}850 \text{ psi (T)} \ \#$$

$$\sigma_{p2} = 1000 - 9849 = -8849 \text{ psi} \cong 8850 \text{ psi (C)} \ \#$$

$$\sigma_{p3} = \sigma_z = 0 \ \#$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{10{,}849 - (-8849)}{2} = 9849 \text{ psi} \cong 9850 \text{ psi} \ \#$$

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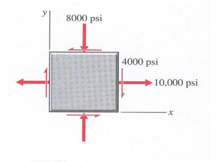
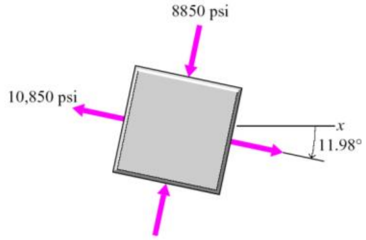
Example Problem 2-11



$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-4000)}{10000-(-8000)} = -0.4444$$

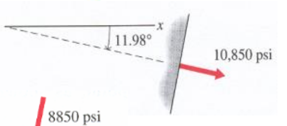
$$2\theta_p = -23.96°, \quad +156.04°$$

$$\theta_p = -11.98°, \quad +78.02° \quad \#$$



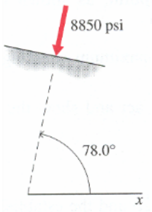
- $\theta_p = -11.98°$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$= 10000\cos^2(-11.98°) + (-8000)\sin^2(-11.98°)$$
$$+ 2(-4000)\sin(-11.98°)\cos(-11.98°)$$
$$= \sigma_{p1} = 10{,}849\ psi \cong 10{,}850\ psi\,(T)$$



- $\theta_p = +78.02°$

$$\sigma_n = 10000\cos^2(78.02°) + (-8000)\sin^2(78.02°)$$
$$+ 2(-4000)\sin(78.02°)\cos(78.02°)$$
$$= \sigma_{p2} = -8849\ \text{psi} \cong -8850\ \text{psi}\,(C)$$



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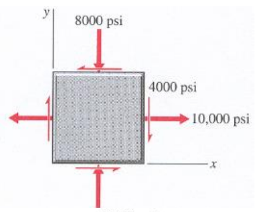
Example Problem 2-11

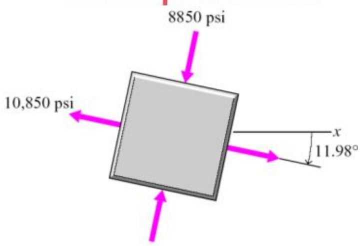


the max. in-plane shear stress

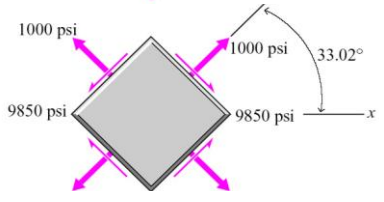
- $\theta_\tau = -11.98° + 45° = +33.02°$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_n = +999.6\ psi \cong 1000\ psi\,(T)$$



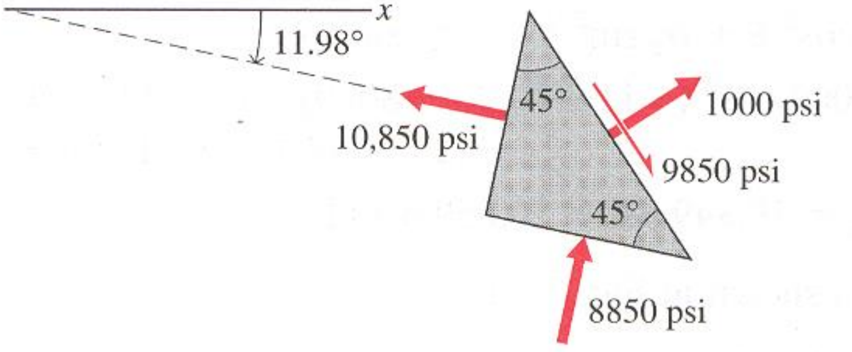
$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= -(10000-(-8000))\sin 33.02°\cos 33.02°$$
$$+ (-4000)(\cos^2 33.02° - \sin^2 33.02°)$$
$$= -9849\ \text{psi} \cong -9850\ \text{psi}$$



(On θ_τ , $\sigma_n = \frac{\sigma_x + \sigma_y}{2}; \tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}$)

p. 93

Example Problem 2-11



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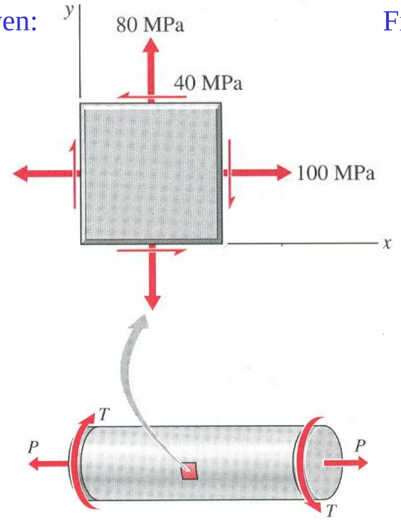
p. 93

Example Problem 2-12

Given:



(a)

Find:

- σ_p , τ_{max} =?
- θ_p , θ_τ = ?
- Show the stresses on a sketch.

σ_x = +100 MPa

σ_y = +80 MPa

τ_{xy} = –40 MPa

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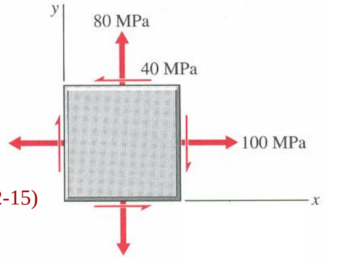
p. 93

Example Problem 2-12



$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{(2-15)}$$

$$= \frac{100+80}{2} \pm \sqrt{\left(\frac{100-80}{2}\right)^2 + (-40)^2}$$

$$= 90 \pm 41.23$$

$$\sigma_{p1} = 90 + 41.23 = +131.23\ \text{MPa} \cong 131.2\ \text{MPa (T)} \qquad \sigma_{p1} \bullet \sigma_{p2} > 0$$

$$\sigma_{p2} = 90 - 41.23 = +48.77\ \text{MPa} \cong 48.8\ \text{MPa (T)}$$

$$\sigma_{p3} = \sigma_z = 0$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{131.23 - 0}{2} = 65.61\ \text{MPa} \cong 65.6\ \text{MPa} \qquad \text{(2-19)}$$

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p. 94

Example Problem 2-12

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-40)}{100-80} = -0.4000 \qquad \text{(2-14)}$$

$$2\theta_p = -75.96°, \quad +104.04°$$

$$\theta_p = -37.98°, \quad +52.02°$$

- $\theta_p = -37.98°$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad \text{(2-12a)}$$

$$= 100\cos^2(-37.98°) + 80\sin^2(-37.98°) + 2(-40)\sin(-37.98°)\cos(-37.98°)$$

$$= \sigma_{p1} = +131.23\ \text{MPa} \cong 131.2\ \text{MPa (T)}$$

- $\theta_p = +52.02°$

$$\sigma_n = 100\cos^2(+52.02°) + 80\sin^2(+52.02°) + 2(-40)\sin(+52.02°)\cos(+52.02°)$$

$$= \sigma_{p2} = +48.77\ \text{MPa} \cong 48.8\ \text{MPa (T)}$$

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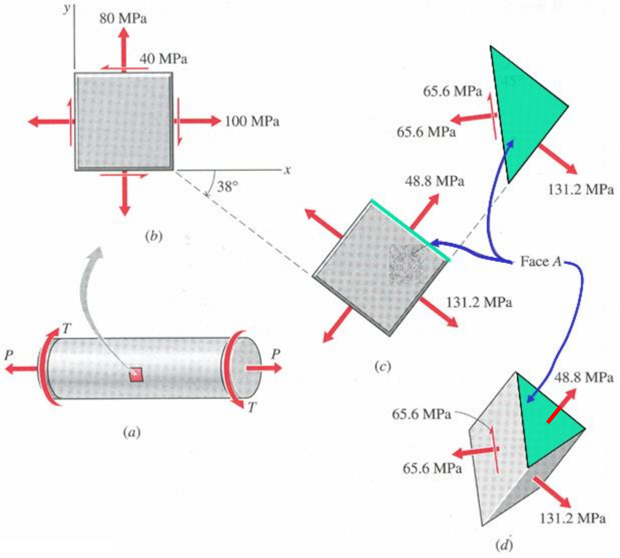
p. 94

Example Problem 2-12



Note:

In *xy*-plane,

$\tau_p = (\sigma_{p1} - \sigma_{p2})/2$

$= (131.2\text{-}48.8)/2$

$= 41.2$ MPa

acting on $\theta = 38° + 45°$

$\tau_p < \tau_{max} = 65.6$ MPa

p. 98

2-11 Mohr's Circle for Plane Stress

Otto Mohr (German engineer, 1835-1918)

$A = 2\theta_p$

$B = 2\theta$

cos(A-B) = cos(A) cos(B) + sin(A) sin(B)

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta = \sigma_{avg} + R\cos(2\theta_p - 2\theta)$$

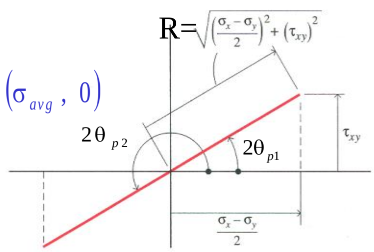
$$\tau_{nt} = -\sin 2\theta \frac{\sigma_x - \sigma_y}{2} + \cos 2\theta\ \tau_{xy} = R\sin(2\theta_p - 2\theta)$$

square sum sin(-B+A) = sin(-B)cos(A) + cos(-B)sin(A)

$$\Rightarrow \left(\sigma_n - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{nt}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Circle : center $(\sigma_n,\ \tau_{nt}) = \left(\frac{\sigma_x + \sigma_y}{2},\ 0\right) = (\sigma_{avg},\ 0)$

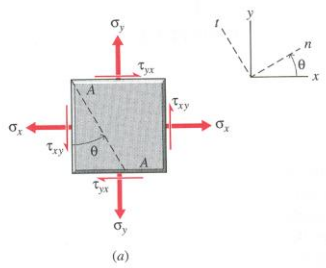
radius $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

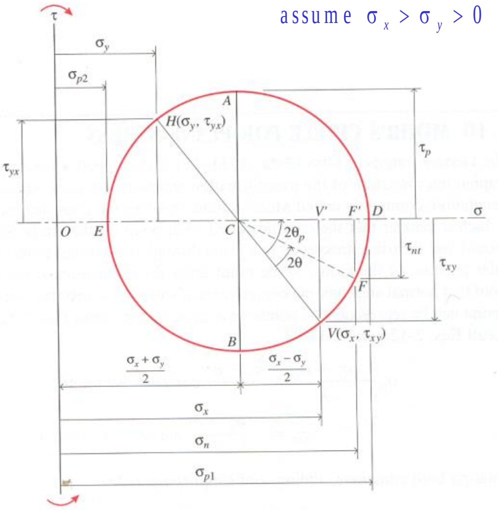


p. 99

2-11 Mohr's Circle for Plane Stress

Shear stress tends to rotate the element **clockwise**





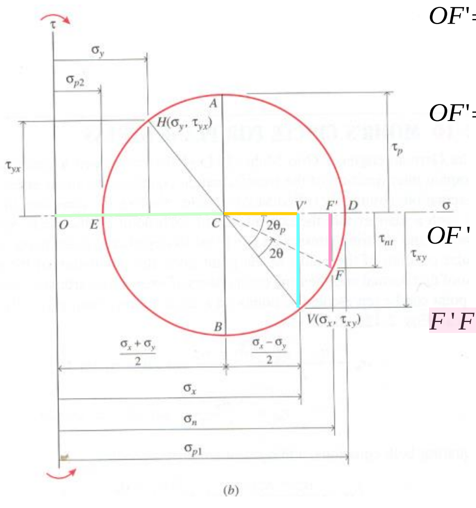
$C : (\sigma_{avg}, 0)$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Shear stress tends to rotate the element **counterclockwise**

p. 100

2-11 Mohr's Circle for Plane Stress



$$OF' = OC + CF\cos(2\theta_p - 2\theta)$$

$$= (\sigma_x + \sigma_y)/2 = \sigma_{avg} \qquad = VV' = \tau_{xy}$$

$$OF' = OC + CV\cos 2\theta_p \cos 2\theta + CV\sin 2\theta_p \sin 2\theta$$

$$= CV' = (\sigma_x - \sigma_y)/2$$

$$OF' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta = \sigma_n$$

$$F'F = CF\sin(2\theta_p - 2\theta)$$

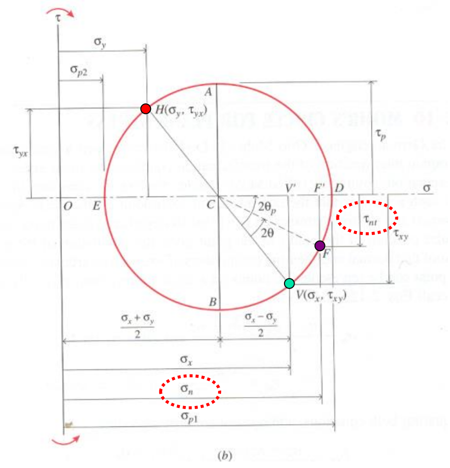
$$= CV\sin 2\theta_p \cos 2\theta - CV\cos 2\theta_p \sin 2\theta$$

$$= V'V\cos 2\theta - CV'\sin 2\theta$$

$$= \tau_{xy}\cos 2\theta - \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \tau_{nt}$$

p. 100

2-11 Mohr's Circle for Plane Stress



$$\sigma_n = \sigma_{avg} + R\cos\left(2\theta_p - 2\theta\right)$$
$$\tau_{nt} = R\sin\left(2\theta_p - 2\theta\right)$$
⇨ Point F

- $\theta = 0^\circ$

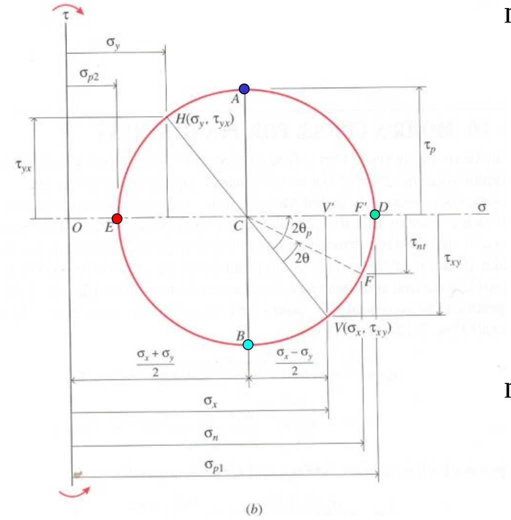
$$\sigma_n = \sigma_{avg} + R\cos 2\theta_p = \sigma_x$$
$$\tau_{nt} = R\sin 2\theta_p = \tau_{xy}$$
⇨ Point V

- $\theta = 90^\circ$

$$\sigma_n = \sigma_{avg} - R\cos 2\theta_p = \sigma_y$$
$$\tau_{nt} = -R\sin 2\theta_p = -\tau_{xy}$$
⇨ Point H

p. 100

2-11 Mohr's Circle for Plane Stress



max. normal stress:

$$\sigma_{p1} = \sigma_{avg} + R\cos(2\theta_p - 2\theta_p)$$
$$= \sigma_{avg} + R$$
$$\tau_{nt} = -R\sin(2\theta_p - 2\theta_p) = 0$$

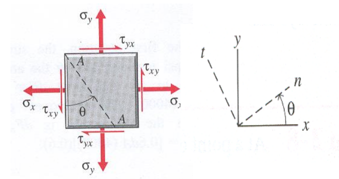
$$\sigma_{p2} = \sigma_{avg} + R\cos(2\theta_p - 2\theta_p - 180°)$$
$$= \sigma_{avg} - R$$
$$\tau_{nt} = -R\sin(2\theta_p - 2\theta_p - 180°) = 0$$

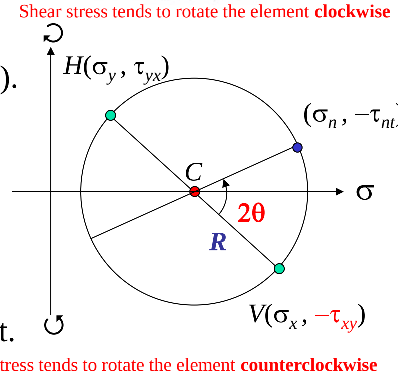
max. in-plane shearing stress

$$\tau_p = R\sin(2\theta_p - 2\theta_p \pm 90°) = \pm R$$

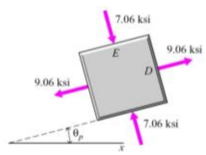
Procedure for Drawing Mohr's Circle

- Choose a set of x-y axes.
- Identify σ_x, σ_y, and τ_{xy} with proper sign.
- Draw a set of σ–τ axes.
- Plot points $V(\sigma_x, -\tau_{xy})$ and $H(\sigma_y, \tau_{yx})$.
- Draw line VH and determine the center C and radius R.
- Draw the circle.
- Use CV as the x-axis ($\theta = 0°$) or the reference line for angle measurement.



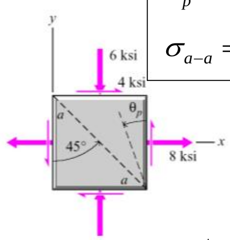


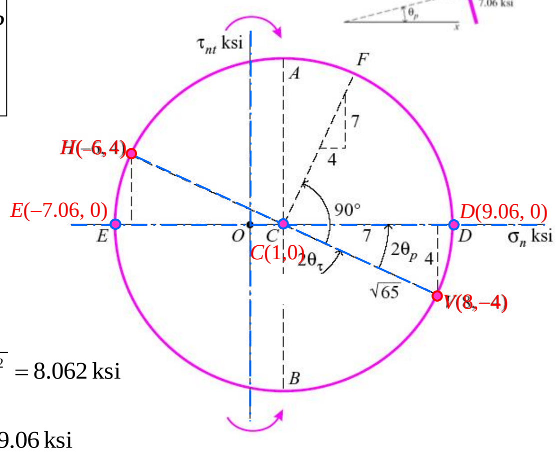
Example Problem 2-13



$\sigma_p = ?\ \tau_{max} = ?\ \theta_p = ?$

$\sigma_{a-a} = ?\ \tau_{a-a} = ?$





$$OC = (\sigma_x + \sigma_y)/2 = (8-6)/2 = 1$$

$$CV = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{7^2 + 4^2} = 8.062 \text{ ksi}$$

$$\sigma_{p1} = OD = OC + CV = 1 + 8.062 = 9.06 \text{ ksi}$$

$$\sigma_{p2} = OE = CE - CV = 1 - 8.062 = -7.06 \text{ ksi}$$

$$\sigma_{p3} = \sigma_z = 0$$

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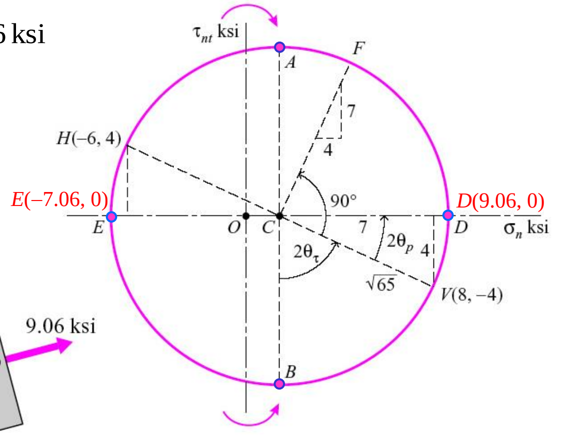
Example Problem 2-13

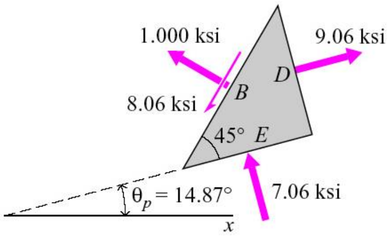
$$\tau_p = \tau_{max} = CA = CB = 8.06 \text{ ksi}$$

$$\sigma_n = OC = 1.000 \text{ psi}$$

$$2\theta_p = \tan^{-1}(4/7) = 29.74°$$

$$\theta_p = 14.87°$$





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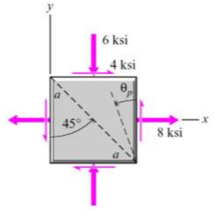
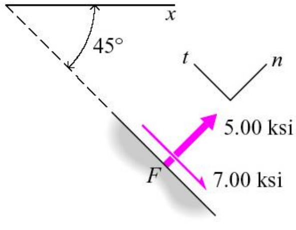
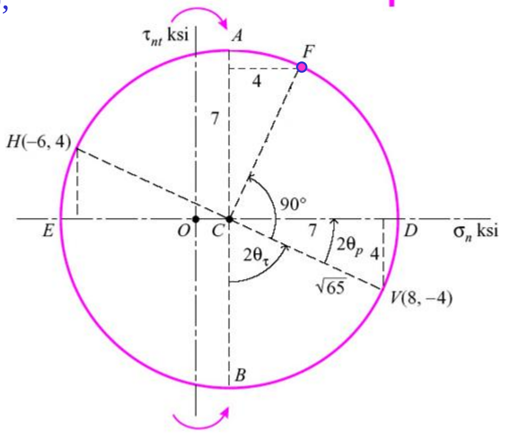
Example Problem 2-13



At plane *a*-*a*, 45° from *V*(8, −4), 90° CCW from *CV*, i.e., *F*(5, 7)





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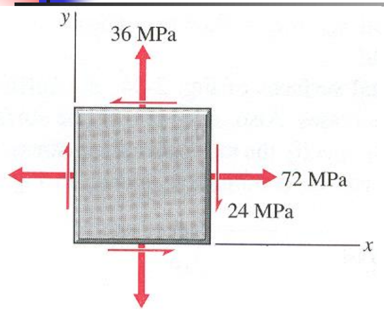
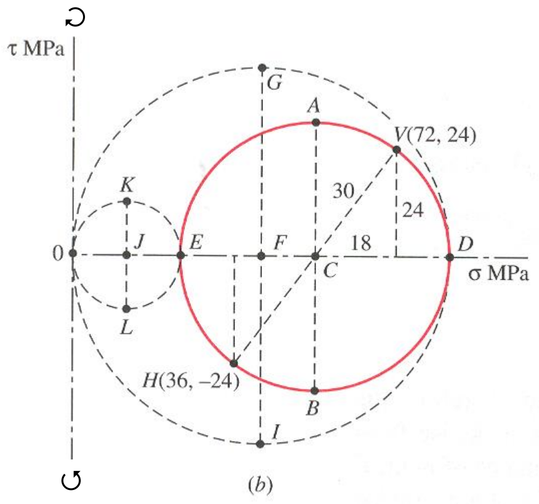
Example Problem 2-14





$\sigma_{p1} = OD = 54 + 30 = 84.0$ MPa (T)

$\sigma_{p2} = OE = 54 - 30 = 24.0$ MPa (T)

$\sigma_{p3} = \sigma_z = 0$

$\tan 2\theta_p = -24/18 = -1.3333$

$\theta_p = -26.57? = 26.57°$ ↻

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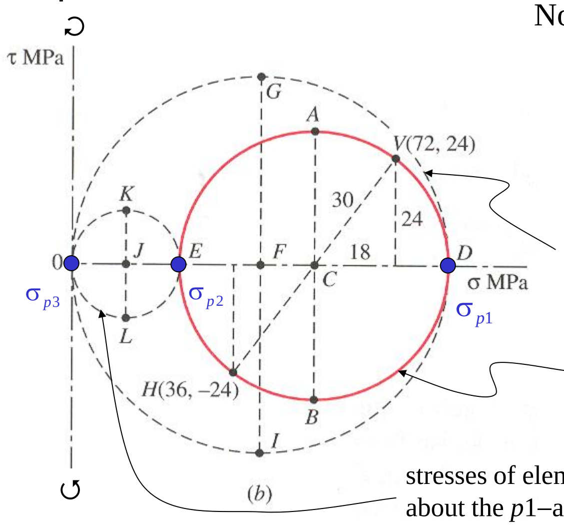
Example Problem 2-14

Note: $\sigma_{p1} > \sigma_{p2} > \sigma_{p3} = 0$

$$\tau_{max} = \frac{\sigma_{p1} - 0}{2} = 42 \text{ MPa}$$

$$\neq \tau_p = \frac{\sigma_{p1} - \sigma_{p2}}{2}$$



stresses of elements rotating about the *p*2–axis

stresses of elements rotating about the *p*3–axis (*z*–axis)

stresses of elements rotating about the *p*1–axis

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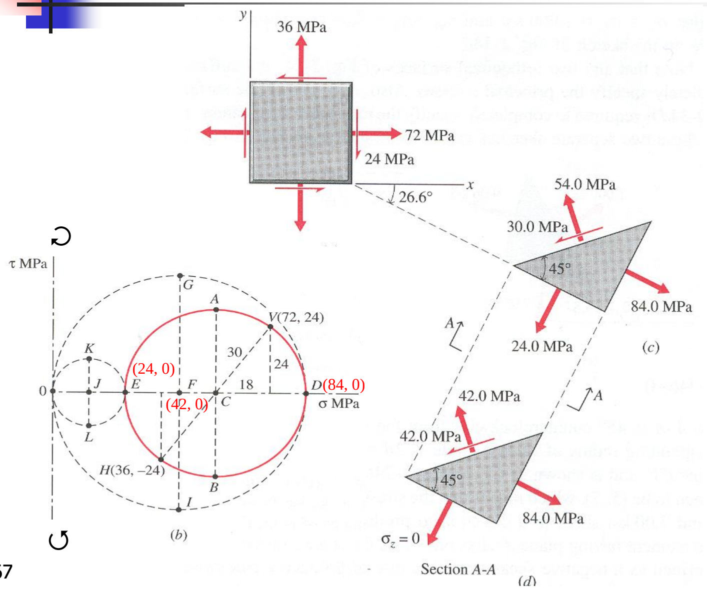
Example Problem 2-14



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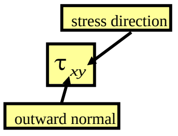
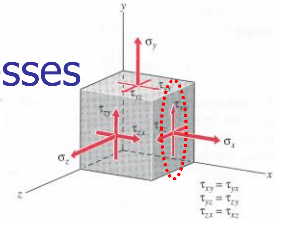
Remarks for sign of shear stresses

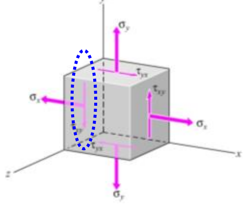
Sign conventions: Positive in the **positive** direction of the j-axis

- positive stresses τ_{ij} or $\tau_{(-i)(-j)}$

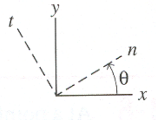
Positive in the **negative** direction of the j-axis

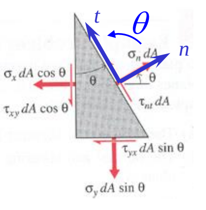




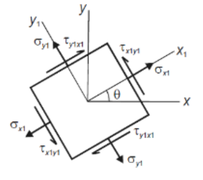


- +θ CCW from +*x*-axis



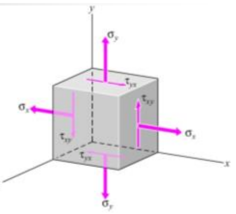


- (*n*, *t*, *z*) axes have the same order as the (*x*,*y*,*z*) axes. Both are right-hand coordinate system.

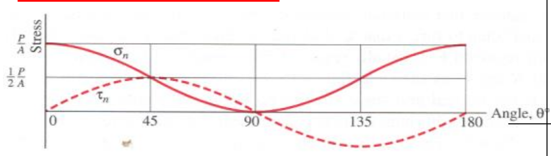


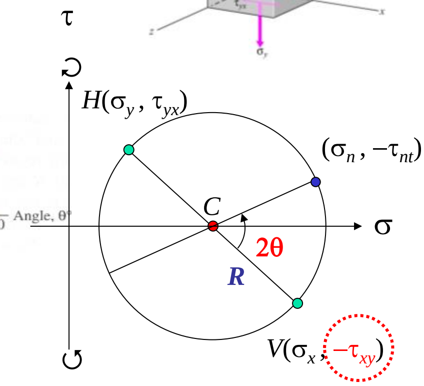
Remarks for sign of shear stresses

$$\tau_{yx} = \tau_{xy} \quad \Rightarrow \quad |\tau_{yx}| = |\tau_{xy}|$$



$$\tau(\theta) = -\tau(\theta + 90°)$$





Shear stress tends to rotate the element **counterclockwise**

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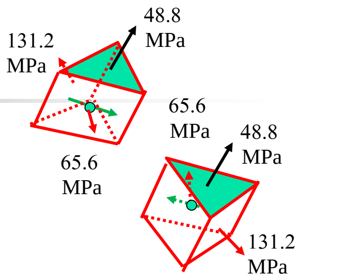
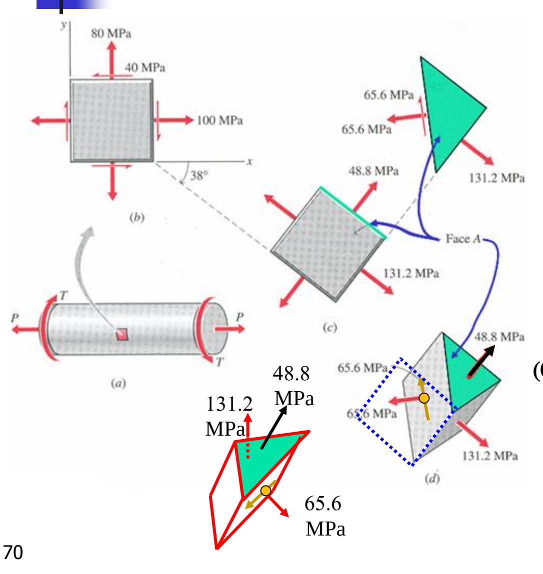
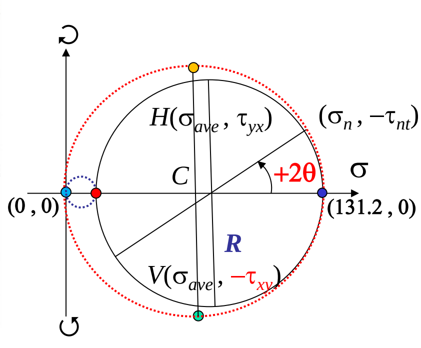
Sign of shear stresses for Example Problem 2-12







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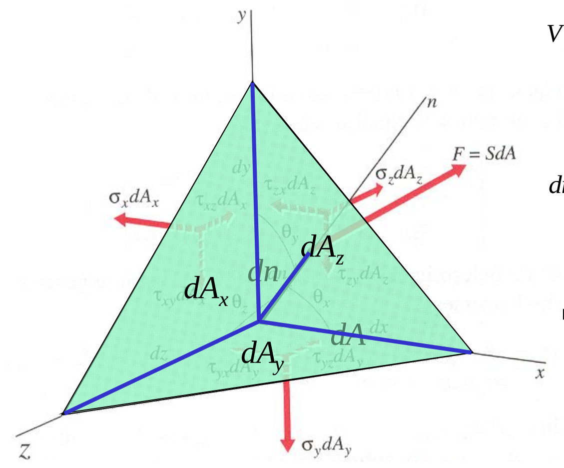
2-12 General State of Stress at a Point



$$V = \frac{1}{3} dn\ dA$$

$$= \frac{1}{3} dx\ dA_x = \frac{1}{3} dy\ dA_y = \frac{1}{3} dz\ dA_z$$

$$dn = dx \cos\theta_x = dy \cos\theta_y = dz \cos\theta_z$$

$$\Rightarrow \quad dA_x = dA \cos\theta_x = dA\, l$$

$$dA_y = dA \cos\theta_y = dA\, m$$

$$dA_z = dA \cos\theta_z = dA\, n$$

l, m, n: direction cosines

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2-12 General State of Stress at a Point

Force equilibrium

$$F_x = S_x dA = \sigma_x dA\, l + \tau_{yx} dA\, m + \tau_{zx} dA\, n$$

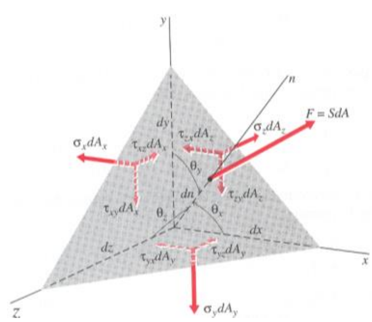
$$F_y = S_y dA = \tau_{xy} dA\, l + \sigma_y dA\, m + \tau_{zy} dA\, n$$

$$F_z = S_z dA = \tau_{xz} dA\, l + \tau_{yz} dA\, m + \sigma_z dA\, n$$

$$S_x = \sigma_x l + \tau_{yx} m + \tau_{zx} n$$

$$S_y = \tau_{xy} l + \sigma_y m + \tau_{zy} n$$

$$S_z = \tau_{xz} l + \tau_{yz} m + \sigma_z n$$



$$\sigma_n = S_x l + S_y m + S_z n$$

$$= \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl$$

$$\tau_{nt} = \pm\sqrt{S^2 - \sigma_n^2}$$

8 Exercises

2-41, 2-54, 2-61, 2-81,
2-84, 2-93, 2-96, 2-119

Appendix

Greek alphabet

Upper case	Lower case	Name	Upper case	Lower case	Name
Α	α	alpha	Ν	ν	nu
Β	β	beta	Ξ	ξ	xi /ksi/ or /zai/
Γ	γ	gamma	Ο	ο	omicron
Δ	δ	delta	Π	π	pi
Ε	ε	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σ	sigma
Η	η	eta	Τ	τ	tau
Θ	θ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	ϕ	phi /fai/
Κ	κ	kappa	Χ	χ	chi /kai/
Λ	λ	lambda	Ψ	ψ	psi /sai/
Μ	μ	mu	Ω	ω	omega