Mechanics of Materials

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Chapter 8

Flexural Loading Beam Deflections

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Flexural Loading Beam Deflections

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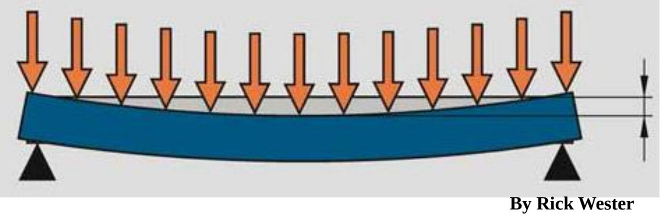
8-1 Introduction

- Deflections are calculated in order to verify that they are within tolerable limits
- The deflection of a beam depends on the stiffness of the material and dimensions of beam as well as on the applied loads and supports



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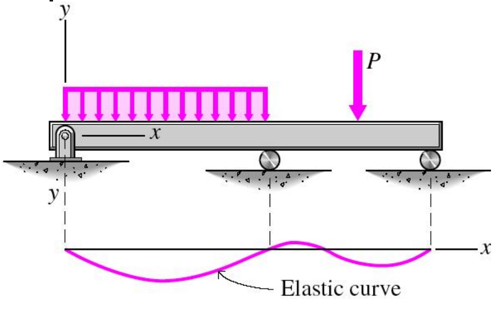
8-2 The Differential Equation of the Elastic Curve



- Straight (horizontal) beam
- Elastic
- **Deflection (vertical displacement) *v***

v: **positive upward**

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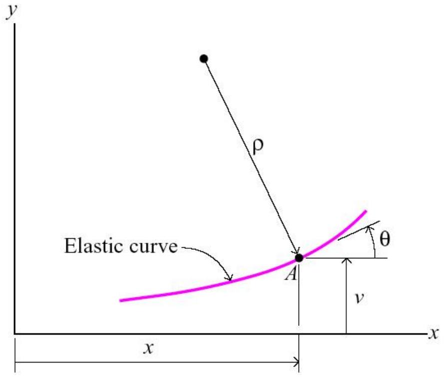
8-2 The Differential Equation of the Elastic Curve



- Elastic curve
- Slope of the curve:

$$\text{slope} = \frac{dv}{dx} = \tan\theta$$

for small slope, $\tan\theta \approx \theta$

$$\boxed{\theta = \frac{dv}{dx}}$$

- Curvature of the curve:

$$\boxed{\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}}}$$

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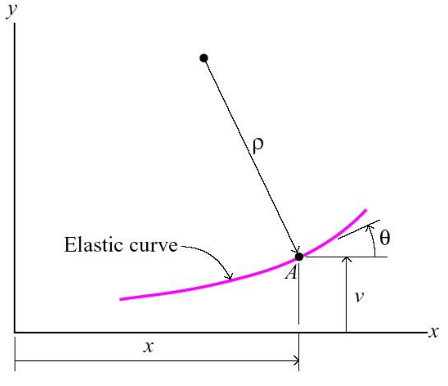
8-2 The Differential Equation of the Elastic Curve

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}}$$



- For the small slope,

$$1 >> \theta^2 = (dv/dx)^2$$

$$\boxed{\frac{1}{\rho} = \frac{d^2v}{dx^2} = \frac{d\theta}{dx}}$$

- Recall Eqs.(7-3) and (7-8)

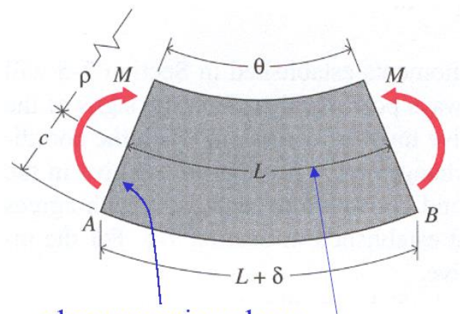
$$\left.\begin{aligned} \sigma_x &= E\varepsilon_x = E\left(\frac{-y}{\rho}\right) \\ \sigma_x &= \frac{-M_r y}{I} \end{aligned}\right\} \quad \frac{1}{\rho} = \boxed{\frac{M_r}{EI} = \frac{d^2v}{dx^2}}$$

$$\Rightarrow \quad \boxed{EI\frac{d^2v(x)}{dx^2} = M_r(x)}$$

Differential equation for the elastic curve

8-2 The Differential Equation of the Elastic Curve

Or alternatively,



- Straight beam
- Linearly Elastic material
- The beam is bent with couples.

$$\theta = \frac{L}{\rho} = \frac{L+\delta}{\rho + c} \Rightarrow 1 = \frac{1+\delta/L}{1+c/\rho}$$

$$\Rightarrow \frac{c}{\rho} = \frac{\delta}{L} = \varepsilon = \frac{\sigma}{E} = \frac{Mc}{EI}$$

$$\Rightarrow \boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

Curvature for M = M(x)

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]^{3/2}}$$

- For most beams, $dy/dx << 1$.

$$\Rightarrow \frac{1}{\rho} = \frac{d^2y}{dx^2}$$

$$\because \frac{1}{\rho} = \frac{M}{EI} \Rightarrow \boxed{EI\frac{d^2y}{dx^2} = M(x)}$$

Differential equation for the elastic curve

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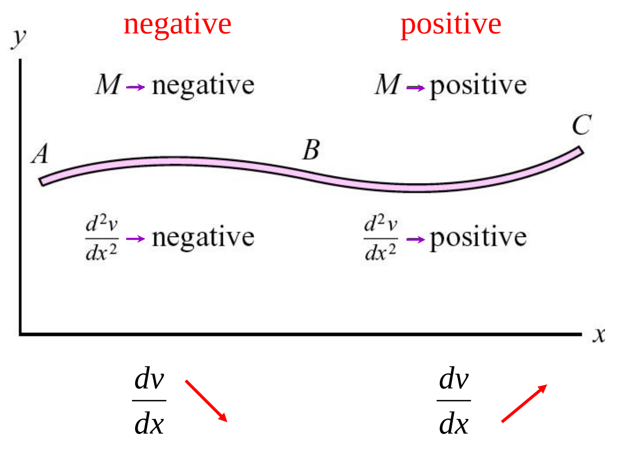
Sign Convention



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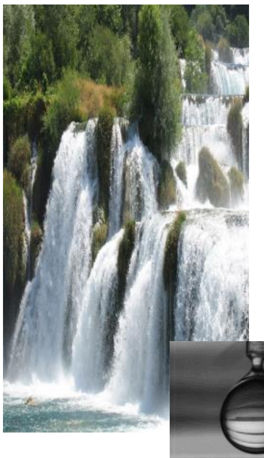
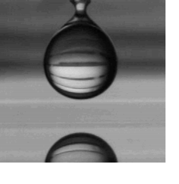
Relation of Physical Quantities and y

- **deflection** $= y$
- **slope** $= \frac{dy}{dx}$
- **moment** $= EI\frac{d^2y}{dx^2}$
- **Shear** $= \frac{dM}{dx} = EI\frac{d^3y}{dx^3}$
- **load** $= \frac{dV}{dx} = EI\frac{d^4y}{dx^4}$





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8-3 Deflection by Integration

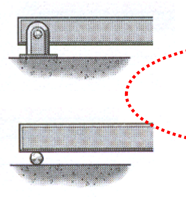
$$EI\frac{d^2y}{dx^2} = M$$

$$EI\frac{d^3y}{dx^3} = V$$

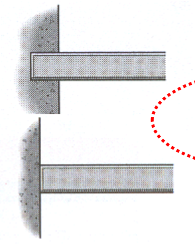
$$EI\frac{d^4y}{dx^4} = w$$

integration → $\boldsymbol{y}$

+ **boundary conditions**



$y = 0$
$d^2y/dx^2 = 0$



$y = 0$
$dy/dx = 0$

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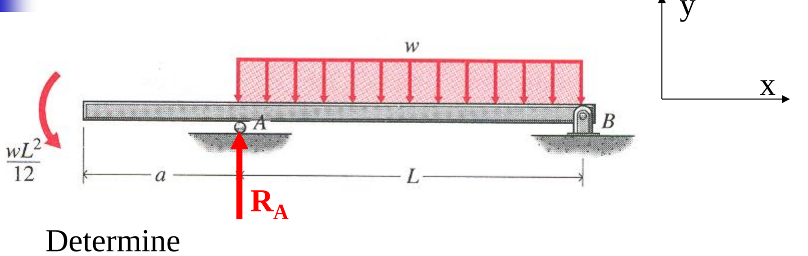
Example Problem 8-2 (I)



Determine
Equation of elastic curve, position of maximum deflection, and its maximum deflection between supports

$$\sum M_B = -R_A(L) + \frac{wL^2}{12} + wL\left(\frac{L}{2}\right) = 0$$

$$\Rightarrow R_A = +\frac{7wL}{12} = \frac{7wL}{12}\uparrow$$

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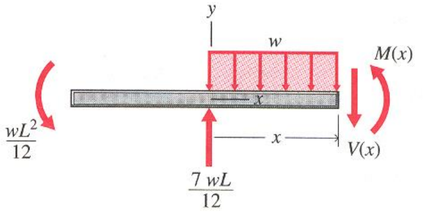
Example Problem 8-2 (II)



$$EI\frac{d^2y}{dx^2} = M(x) = \frac{7wL}{12}x - \frac{wL^2}{12} - wx\left(\frac{x}{2}\right) \qquad \text{for} \qquad 0 \le x \le L$$

$$EI\frac{dy}{dx} = \frac{7wL}{24}x^2 - \frac{wL^2}{12}x - \frac{w}{6}x^3 + C_1$$

$$EIy = \frac{7wL}{72}x^3 - \frac{wL^2}{24}x^2 - \frac{w}{24}x^4 + C_1x + C_2$$

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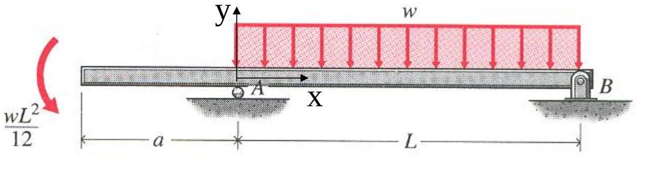
Example Problem 8-2 (III)



$$EIy = \frac{7wL}{72}x^3 - \frac{wL^2}{24}x^2 - \frac{w}{24}x^4 + C_1x + C_2$$

B.C. at A: $x = 0, y = 0. \quad C_2 = 0$

at B: $x = L, \; y = 0. \quad C_1 = -\frac{wL^3}{72}$

$$y = -\frac{w}{72EI}\left(3x^4 - 7Lx^3 + 3L^2x^2 + L^3x\right)$$

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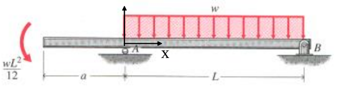
Example Problem 8-2 (IV)



- Maximum deflection between supports

$$\frac{dy}{dx} = -\frac{w}{72EI}\left[12x^3 - 21Lx^2 + 6L^2x + L^3\right] = 0$$

$$x = -0.1162L\,,\; x = 0.541L\,,\; x = 1.325L$$

Maximum deflection (between supports) occurs at $x = 0.541L$

$$y = -\frac{w}{72EI}\left(3x^4 - 7Lx^3 + 3L^2x^2 + L^3x\right)\Big|_{x=0.541L}$$

$$= -7.88\left(10^{-3}\right)\frac{wL^4}{EI} = 7.88\left(10^{-3}\right)\frac{wL^4}{EI}\downarrow$$

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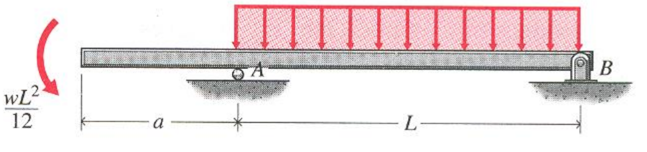
Remarks



- If the deflection of beam to the left of support A is also required
  - Derive $M(x)$ (or $V(x)$, $w(x)$) for that portion
  - Integrate the differential equation.
  - Apply the **matching condition at support A**:

$y(A^-) = y(A^+)$, $y'(A^-) = y'(A^+)$

3 Exercises

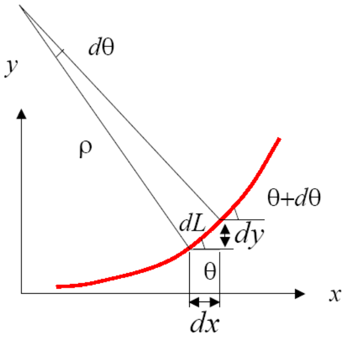
- **8-30, 8-35, 8-44.**

Appendix

Derivation of radius of curvature (I)

$$\theta = \tan^{-1}\frac{dy}{dx} = \tan^{-1} y' = y' - \frac{1}{3}y'^3 + \frac{1}{5}y'^5 - \cdots$$

$$(\tan^{-1} z = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \cdots)$$



$$\theta + d\theta = \tan^{-1}\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}dx + \cdots\right)$$

$$= y' + y''dx - \frac{1}{3}(y' + y''dx)^3 + \cdots$$

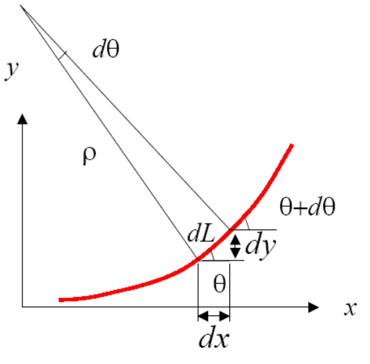
$$\cong y' + y''dx - (\frac{1}{3}y'^3 + y'^2 y''dx + y'y''^2 dx^2 + \frac{1}{3}y''^3 dx^3)$$

Neglecting higher order terms,

⇨ $d\theta \cong y''(1 - y'^2)dx$

Derivation of radius of curvature (II)

$$d\theta \cong y''\left(1 - y'^2\right)dx$$



$$\rho d\theta = dL = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{dx^2 + \left(\frac{dy}{dx}dx\right)^2} = dx\sqrt{1 + y'^2}$$

$$\rho = \frac{dL}{d\theta} = \frac{\sqrt{1 + y'^2}}{y''\left(1 - y'^2\right)} = \frac{\sqrt{1 + y'^2}}{y''}\left(1 + y'^2 + \cdots\right)$$

$$\cong \frac{\left(1 + y'^2\right)^{3/2}}{y''}$$