Chapter 2
Signal Detection and Image Reconstruction Through Fourier Transform

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Transverse magnetization ⟶ MRI Signal

$$\vec{M}(\vec{r},t) \longrightarrow \text{V(t)}$$

$$\Phi(t) = \int_{obj} \vec{B}_r(\vec{r}) \cdot \vec{M}(\vec{r},t) d\vec{r}$$

$\vec{B}_r$ Reception sensitivity: The magnetic field at $\vec{r}$ induced by a hypothetical unit current

$$\text{V(t)} = \frac{-\partial\Phi(t)}{\partial t} = -\frac{\partial}{\partial t}\int_{obj} \vec{B}_r(\vec{r}) \cdot \vec{M}(\vec{r},t) d\vec{r}$$

$$= -\int_{obj}\left[B_{r,x}(\vec{r})\frac{\partial M_x}{\partial t} + B_{r,y}\frac{\partial M_y}{\partial t}\right]d\vec{r} \qquad \left(\frac{\partial M_z}{\partial t} \sim 0\right)$$

Let $B_{r,x} = \left|B_{r,xy}(\vec{r})\right| Cos\phi_r(\vec{r})$ $\qquad B_{r,xy} \equiv B_{r,x} + iB_{r,y}$, $M_{xy} = M_x + iM_y$

$$B_{r,y} = \left|B_{r,xy}(\vec{r})\right| Sin\phi_r(\vec{r})$$

$$M_x(\vec{r},t) = \left|M_{xy}(\vec{r},0)\right| e^{-t/T_2(\vec{r})} Cos\left[-w(\vec{r})t + \phi_e(\vec{r})\right]$$

ϕ_e = initial phase

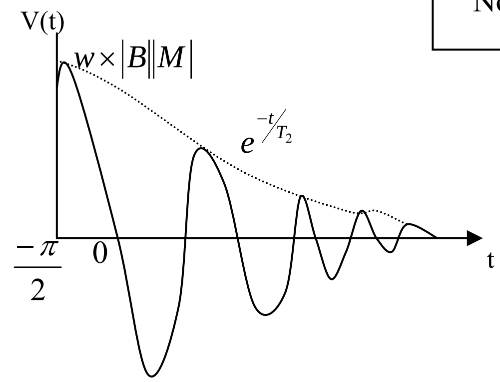
$$M_y(\vec{r},t) = \left|M_{xy}(\vec{r},0)\right| e^{-t/T_2(\vec{r})} Sin\left[-w(\vec{r})t + \phi_e(\vec{r})\right]$$

$$\frac{\partial M_x}{\partial t} = +w(\vec{r})\left|M_{xy}(\vec{r},0)\right| e^{-t/T_2(\vec{r})} Sin\left[-w(\vec{r})t + \phi_e(\vec{r})\right] - \frac{1}{T_2(\vec{r})}\left|M_{xy}\right| e^{-t/T_2} Cos\left[-w(\vec{r})t + \phi_e(\vec{r})\right]$$

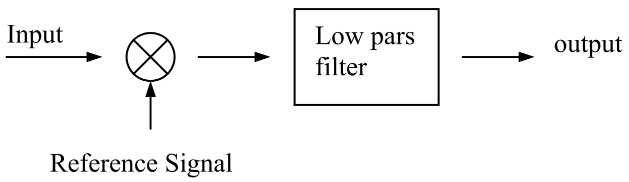
$$\frac{\partial M_y}{\partial t} = -w(\vec{r})\left|M_{xy}(\vec{r},0)\right|e^{-t/T_2}Cos\left[-w(\vec{r})t+\phi_e(\vec{r})\right]-\frac{1}{T_2}\left|M_{xy}\right|e^{-t/T_2}Sin\left[-w(\vec{r})t+\phi_e(\vec{r})\right]$$

$$V(t)=\int_{obj} w(\vec{r})\left|B_{r,xy}(\vec{r})\right|\left|M_{xy}(\vec{r},0)\right|e^{-t/T_2}Cos\left[-wt+\phi_e-\phi_r+\frac{\pi}{2}\right]d\vec{r}$$

Note: $Sin(\alpha-\beta)=Sin\alpha Cos\beta - Cos\alpha Sin\beta$



PSD: phase-sensitive detection or signal demodulation method



$V(t)\times 2Cos(w_0 t)$ then low pass filter

$$S_R = V_{psd}(t) = \int_{obj} \underbrace{w(\vec{r})}_{\approx\, w_0}\left|B_{r,xy}(\vec{r})\right|\left|M_{xy}(\vec{r},0)\right|e^{-t/T_2}Cos\left[\underbrace{-wt+w_0t}_{-\Delta wt}+\phi_e-\phi_r+\frac{\pi}{2}\right]d\vec{r}+$$

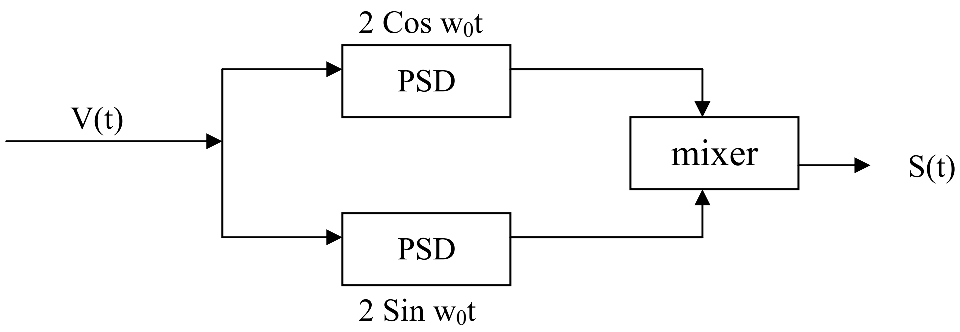
$(\Delta w = w - w_0)$

$$\underbrace{\int\cdots Cos\left[-wt-w_0t-\phi_e+\phi_r\frac{-\pi}{2}\right]dr}_{\text{filtered}}$$

Note: $2Cos\alpha Cos\beta = Cos(\alpha-\beta)+Cos(\alpha+\beta)$
$2Cos\alpha Sin\beta = Sin(\beta-\alpha)+Sin(\beta+\alpha)$

Quadrature detection

Use another PSD : V(t) Ｘ 2 Sin (w_0t)



$$S_I = V_{psd}(t) = w_0 \int_{obj} |B_{r,xy}||M_{xy}| e^{-t/T_2} Sin\left[-\Delta wt + \phi_e - \phi_r + \frac{\pi}{2}\right] d\bar{r}$$

$$S(t) = S_R(t) + iS_I(t)$$

$$= w_0 \int_{obj} |B_{r,xy}||M_{xy}| e^{-i[\Delta wt - \phi_e + \phi_r - \pi/2]} e^{-t/T_2} d\bar{r} \,,$$

Omit $w_0 e^{-i\pi/2}$

Let $B^* = |B| e^{-i\phi_k}$

$M = |M| e^{i\phi_e}$

$$= \int_{obj} B^*_{r,xy}(\bar{r}) M_{xy}(\bar{r},0) e^{-i\Delta wt} e^{-t/T_2} d\bar{r}$$

$$= \int_{obj} M_{xy}(\bar{r},0) e^{-i\Delta wt} e^{-t/T_2} d\bar{r}$$, assume $B^*_{r,xy}(\bar{r})$ = homogeneous reception filed

If a gradient is applied:

$-i\Delta wt = -i\gamma \Delta Bt$ if $B = B_0 + \Delta B(\bar{r})$ is static

$= -i\gamma \int_0^t \Delta B(\bar{r},\tau) d\tau$ if ΔB is time-varying

$$\therefore S(t) = \int_{obj} M_{xy}(\bar{r},0) e^{-i\gamma \Delta Bt} e^{-t/T_2} d\bar{r} = \int_{obj} M_{xy} e^{-iwt} e^{-t/T_2} d\bar{r}$$

Let $dM_{xy} = M_{xy}(\vec{r})d\vec{r} = \rho(w)dw$ isochromatic bulk magnetization

$$S(t) = \int_{-\infty}^{\infty} \rho(w) e^{-t/T_2} e^{-iwt} dw$$

$$= \int_{-\infty}^{\infty} \rho(\vec{r}) e^{-t/T_2} e^{-i\gamma \vec{g}\cdot\vec{r}t} d\vec{r} \qquad \left(\text{apply a gradient } \Delta B = \vec{g}\cdot\vec{r}\right)$$

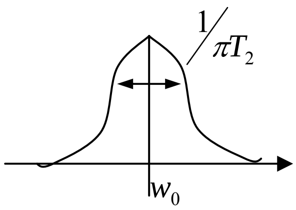
$$S(\vec{k}) = \int_{-\infty}^{\infty} \rho(\vec{r}) e^{-t/T_2} e^{-2\pi i \vec{k}\vec{r}} d\vec{r} \qquad \left(\vec{k} = \gamma \vec{g}t = \frac{\gamma}{2\pi}\int_0^t \vec{g}(\tau)d\tau\right)$$

If no gradient is applied, $w = w_0$

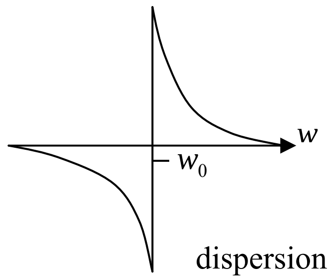
$S(t) = Ae^{-t/T_2} e^{-iw_0 t} \qquad t \geq 0$

$\hat{s}(w) = \mathbf{F}_t \{S(t)\}$

$$\int_0^w Ae^{-t/T_2} e^{-iw_0 t} e^{-iwt} dt = \frac{AT_2}{1 + T_2^2 (w + w_0)^2} - i\frac{AT_2^2 (w + w_0)}{1 + T_2^2 (w + w_0)^2}$$



Lorentzian



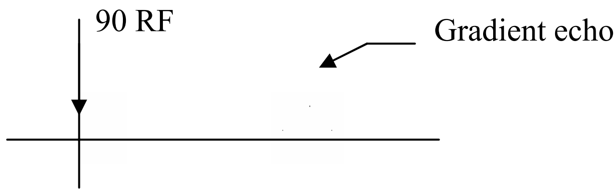
dispersion

Now, let's consider how to spatially encode MR signals.
Suppose a 2-D object $\rho(x, y)$

$$S(k_x, k_y) = \iint \rho(x, y) e^{-2\pi i (k_x x + k_y y)} dxdy$$

How do we get such $S(k_x, k_y)$?

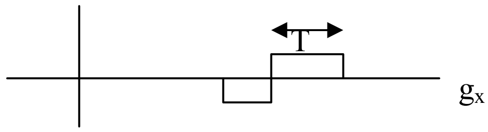
I. Let's apply a gradient g_x



$w(x) = \gamma g_x x$

Collect signals during T:

$$\begin{cases} k_x = \bar{\gamma} g_x t \\ k_y = 0 \end{cases} \qquad k_x: -\bar{\gamma} g_x \frac{T}{2} \sim \bar{\gamma} g_x \frac{T}{2}$$



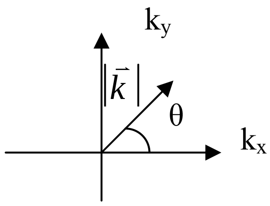
$$\therefore S(k_x, k_y) = \int_x \left(\int_y \rho(x,y)dy \right) e^{-2\pi i k_x x} dx = \mathrm{F}\left[\int \rho(x,y)dy \right]$$

$$\mathbf{F}^{-1}\left[S(k_x k_y) \right] = \int_y \rho(x,y)dy = P_y(x)$$ "projection onto x-axis"

Note: 1) $\vec{k} = \frac{1}{2\pi}\gamma\int_0^t \vec{g}(\tau)d\tau = \frac{1}{2\pi}\gamma\vec{g}t\ \bar{\gamma}$

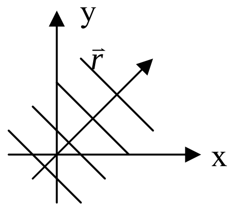
$\phi = \vec{k}\cdot\vec{r}$ $\qquad \vec{k}$ in the unit of L^{-1}

2) $|\vec{k}|$: wave number



θ : wave direction

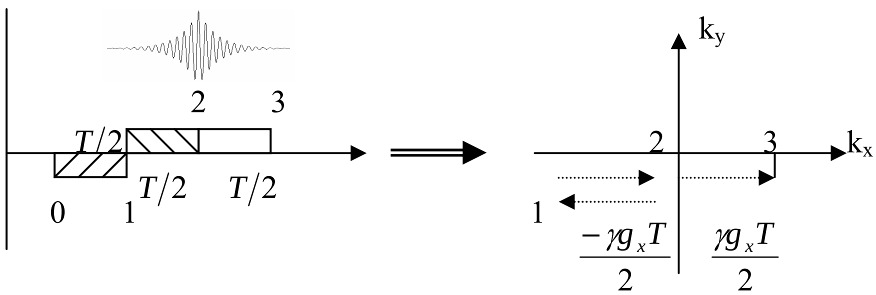
$e^{-i\vec{k}\cdot\vec{r}}$



A 2D-wave

3) $s(\vec{k}) \leftrightarrow \rho(\vec{r})$ Fourier pair

4) k- trajectory: the temporal order of $s(\vec{k})$ that one allocated in k-space

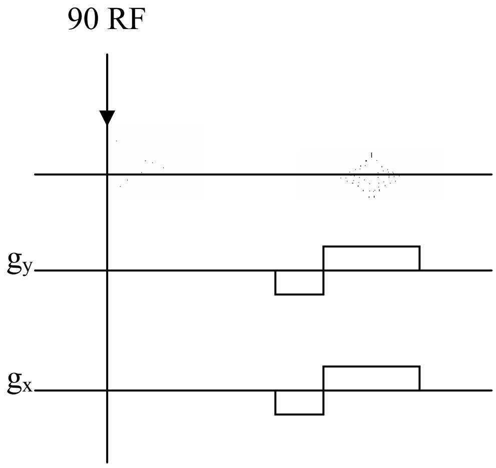


"pre-phasing" =

* echo align w/ center k
* symmetric acquisition

→ No dispersion in Im, no broadening after FT

II. If we apply g_x & g_y simultaneously

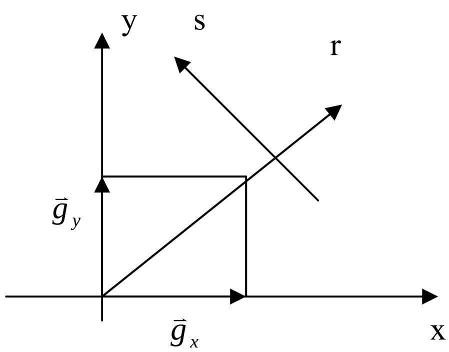


$$S(k_x k_y) = \int_x \int_y \rho(x,y) e^{-2\pi i (k_x x + k_y y)} dxdy$$

$$= \int_s \int_r \rho(s,r) e^{-2\pi i (k_s s + k_r r)} dsdr$$

$$= \int_r \left(\int_s \rho(s,r) ds \right) e^{-2\pi i k_r r} dr$$

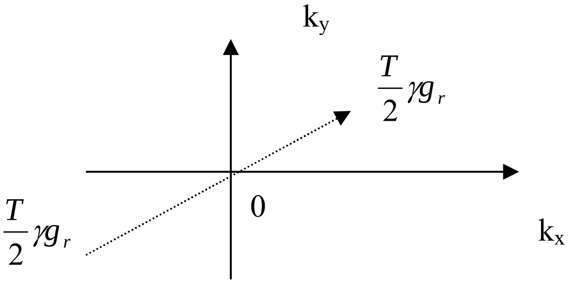
$$= \mathbf{F}_{\mathrm{r}} \left[\int_s \rho(s,r) ds \right]$$



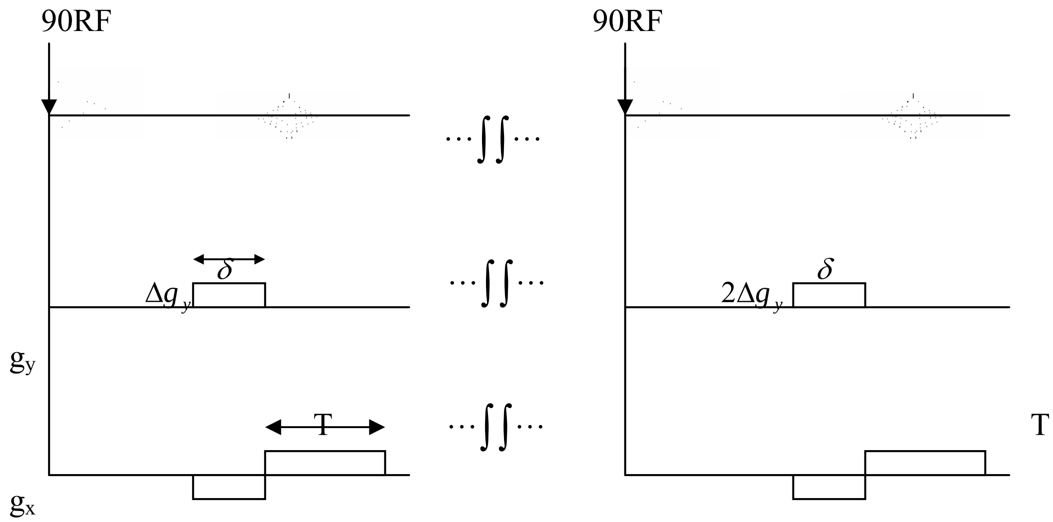
$\int_s \rho(s,r) ds =$ projection of $\rho(s,r)$ along r

∴ only get a line in k-space

$$k_r = -\frac{T}{2}\gamma g_r \to \frac{T}{2}\gamma g_r$$



III. Spin Warp



Repeat for say 128 excitations

$$S(k_x,k_y)=\int_x\int_y \rho(x,y)\, e^{-i\gamma\left(n\Delta g_y \delta_y y + g_x t x\right)}\, dxdy$$

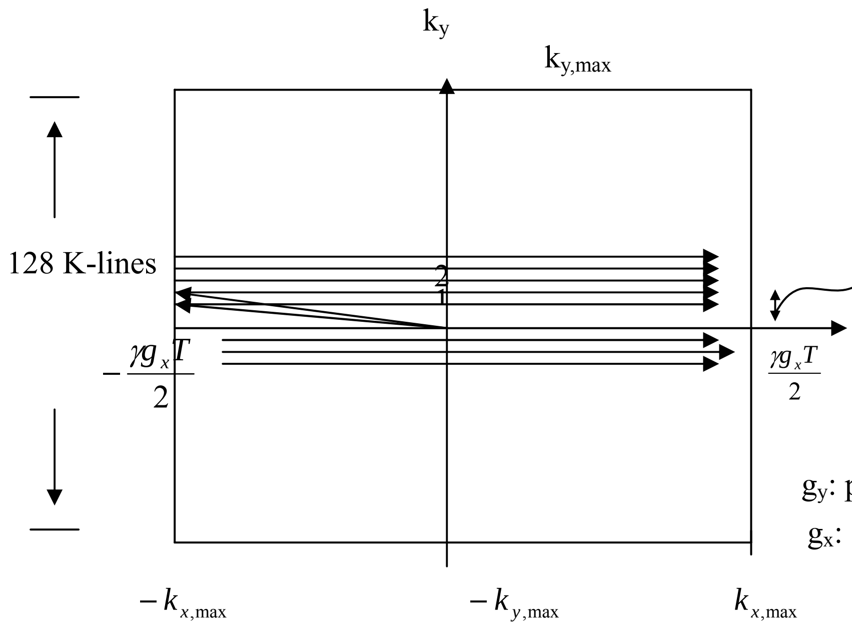
discrete increment (n)

fixed δ_y

continuous (t)

fixed g_x

K-space:



g_y: phase encoding

g_x: frequency encoding

Take 2D F T of $S(k_x,k_y)$ to get $\rho(x,y)$!

(1) Discretization:

digital sampling of $S(k_x, k_y)$ occurs in read-out break k_x into $\gamma g_x\left(m\frac{T}{M}-\frac{T}{2}\right)$, m=0,…..M.

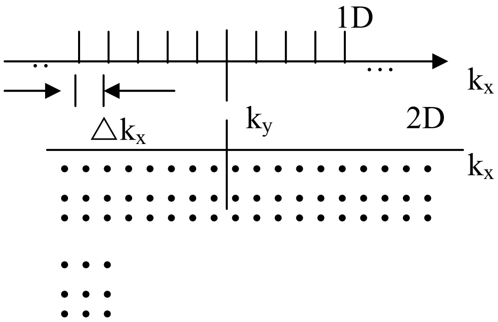
So what we actually get is a discretized $S_s(m\Delta k_x, n\Delta k_y)$

$\gamma n\Delta g_y \delta$

γg_x m Dwell

This is equivalent to :

$$S_s(m\Delta k_x, n\Delta k_y) = S(k_x, k_y)\cdot comb\left(\frac{k_x}{\Delta k_x}\right)\cdot comb\left(\frac{k_y}{\Delta k_y}\right)$$



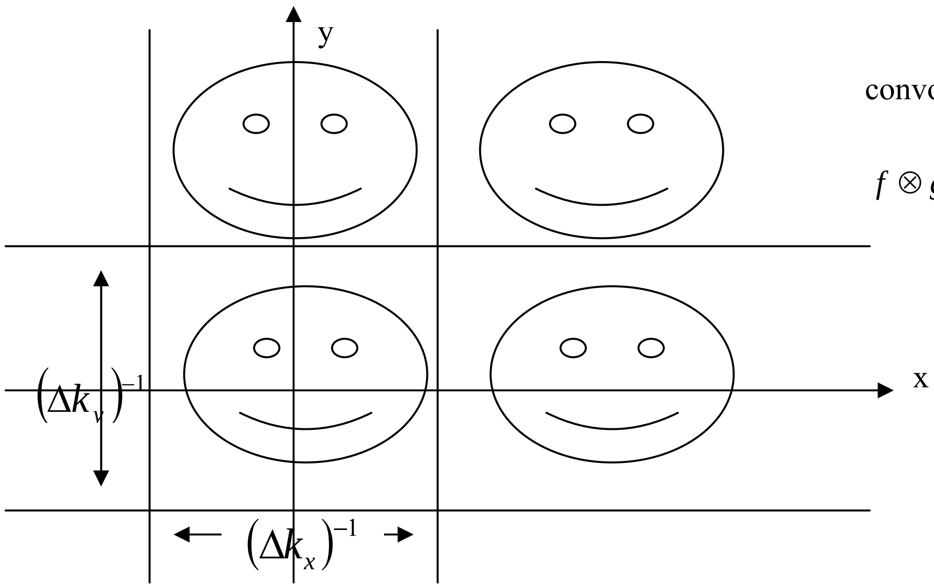
where $comb\left(\frac{k_x}{\Delta k_x}\right) \equiv \sum_{m,n=-\infty}^{\infty}\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$

⇓ F. T.

$$\rho_s(x,y) = \rho(x,y)\otimes \Delta k_x \Delta k_y comb(\Delta k_x x) comb(\Delta k_y y)$$

$$= \rho(x,y)\otimes \sum_{m,n=-\infty}^{\infty}\delta\left(x-\frac{m}{\Delta k_x}, y-\frac{n}{\Delta k_y}\right)\Delta k_x \Delta k_y$$

= a replicated array of $\rho(x,y)$



convolution

$$f \otimes g \equiv \int_{-\infty}^{\infty} f(u)g(v-u)du$$

We can isolate the central one （m ＝ 0, n ＝ 0）with a filter

H（x, y） ＝ rect(Δk_x x) rect(Δk_y y)

so that $\rho(x,y) = \rho_s(x,y)H(x,y)$

Therefore, the close form expression for $S(k_x,k_y)$ can be worked out as follows:

$S(k_x,k_y) = \mathbf{F}\{\rho(x,y)\} =$ $\mathbf{F}\{\rho_s(x,y)\mathbf{H}(\text{x,y})\}$

$\Longrightarrow$ $S(k_x,k_y) = \mathbf{F}[\rho_s(x,y)] \otimes \mathbf{F}\ [H(x,y)]$

$$= \ S(k_x,k_y) comb\left(\frac{k_x}{\Delta k_x}\right) comb\left(\frac{k_y}{\Delta k_y}\right) \otimes \mathbf{F}\left[rect(\Delta k_x x) rect(\Delta k_y y)\right]$$

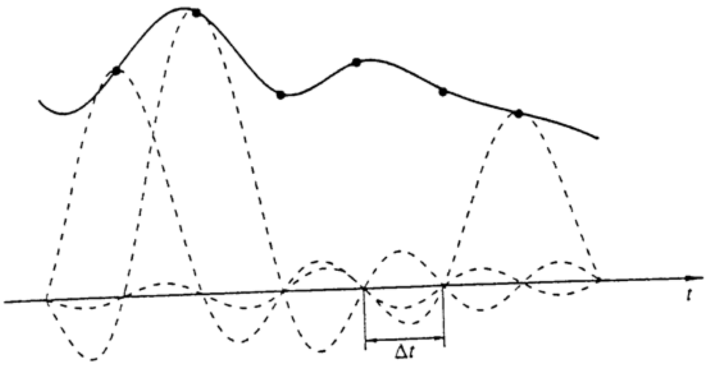
$$= \ \Delta k_x \Delta k_y \sum_{m,n=-\infty}^{\infty} S(k_x,k_y)\delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \otimes$$

$$\frac{1}{\Delta k_x \Delta k_y} Sinc\left(\frac{k_x}{\Delta k_x}\right) Sinc\left(\frac{k_y}{\Delta k_y}\right)$$

$$= \sum_{m,n=-\infty}^{\infty} S(m\Delta k_x, n\Delta k_y) Sinc\left(\frac{k_x - m\Delta k_x}{\Delta k_x}\right) Sinc\left(\frac{k_y - n\Delta k_y}{\Delta k_y}\right)$$

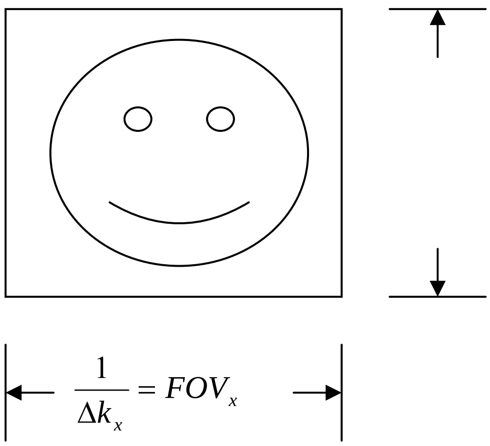
meaning: The sum of the weighted $S(m\Delta k_x, n\Delta k_y)$ with the weighting factor

being $Sinc\left(\frac{k_x}{\Delta k_x}\right) Sinc\left(\frac{k_y}{\Delta k_y}\right)$ that is shifted by $\{m\Delta k_x, n\Delta k_y\}$



" sinc interpolation"
Also used in EPI
rigridding of uniform
time-domoin data
sampling

Let's go back to $\rho_s(x, y)$, a replica of $\rho(x, y)$ the condition that the replica don't overlap:



$$\frac{1}{\Delta k_y} = FOV_y$$

$$\frac{1}{\Delta k_x} = FOV_x$$

$$FOV_x = \frac{1}{\Delta k_x} = \frac{2\pi}{\gamma g Dwell}$$ ← Nyquist criteria!

(1) Nyquist frequency $= \frac{1}{Dwell}$

(2) Nyquist frequency $=$ 2 × highest frequency of a sampled function

$$= 2 \times \left(\gamma g_x \frac{FOV}{2}\right)$$

$$= \gamma g_x \, FOV \qquad \text{“bandwidth”}$$

$$\therefore \quad FOV = \frac{2\pi}{\gamma g_x Dwell} = \frac{1}{\Delta k_x}$$

or FOV = $(\gamma\, g_x\, \text{Dwell})^{-1}$

Therefore, given a fixed g_x, Dwell must be $\leq \dfrac{2\pi}{\gamma g_x FOV}$ or $\leq \dfrac{1}{bandwidth}$

＊ Resolution $\equiv \dfrac{FOV}{m} = \dfrac{2\pi}{\gamma g_x Dwell \cdot m} = \dfrac{2\pi}{\gamma g_x T} = \dfrac{1}{2k_{\max}} = \dfrac{1}{FOV_k}$

$$\text{Dwell} = \frac{2\pi}{\gamma g_x FOV} = \frac{1}{\text{Sweep bandwidth}}$$

$$T = \frac{2\pi}{\gamma g_x (\Delta x)} = \frac{1}{\text{pixel bandwidth}}$$