Mechanics of Materials

Chapter 2

ANALYSIS OF STRESS: CONCEPTS & DEFINITIONS

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ANALYSIS OF STRESS: CONCEPTS & DEFINITIONS

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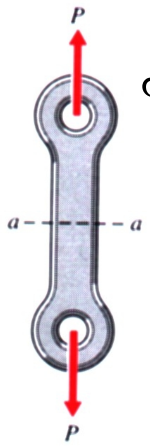
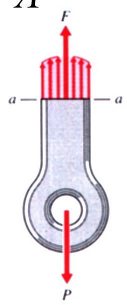
2-2 Normal Stress Under Axial Loading

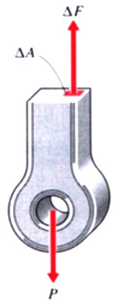
Definition: Stress $= \dfrac{\text{Force}}{\text{Area}}$

intensity of force



$$\sigma_{avg} = \frac{F}{A}$$





$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

σ: "+": tensile stress
"–": compressive stress

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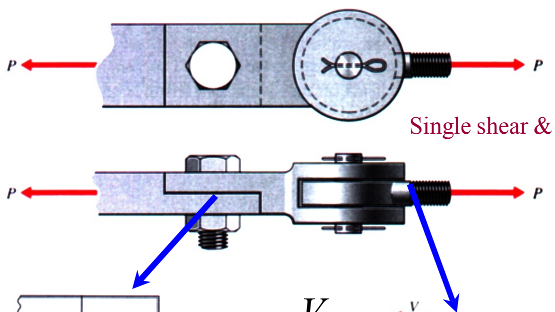
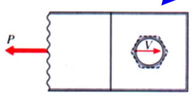
2-3 Shearing Stress in Connections

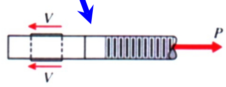
- Connections: rivets, bolts, pins, nails, welds



Single shear & double shear



$$\tau_{avg} = \frac{V}{A}$$



$$\tau = \lim_{\Delta A \to 0} \frac{\Delta V}{\Delta A}$$

(Note: The shear stress τ **cannot** be uniformly distributed over A.)

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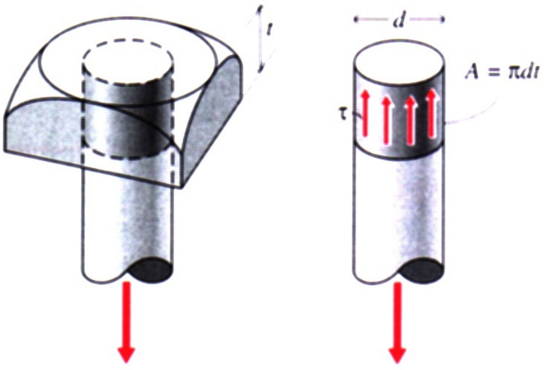
2-3 Shearing Stress in Connections

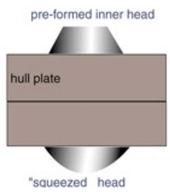
- Punching Shear

periphery area = πdt



(a) (b)

http://thcarmor.theshoppe.com/images/rivet.jpg



www.mrs.org/microworld/ titanic.html

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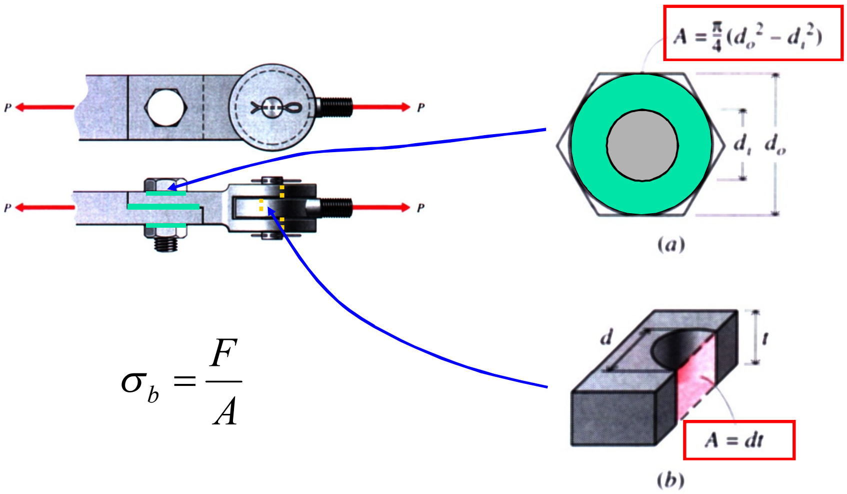
2-4 Bearing Stress

- Bearing stress: **compressive normal stress**



$$\sigma_b = \frac{F}{A}$$

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2-5 Units of Stress

- USCS: United State Customary System (British units)
- SI: International System of Units
- Unit of stress: force per unit area (intensity of force)
  - psi: **p**ound per **s**quare **i**nch
  - ksi: **k**ilo pound per **s**quare **i**nch
    - pound (lb_f): 1 slug · 1 ft/s^2
    - kip: **ki**lo **p**ound
  - Pa: Pascal, Newton per square meter
    - N: Newton, 1 kg · 1 m/s^2 = 0.2248 lb_f
  - MPa: mega (10^6) Newton per square meter
    (1 bar = 0.1 MPa)

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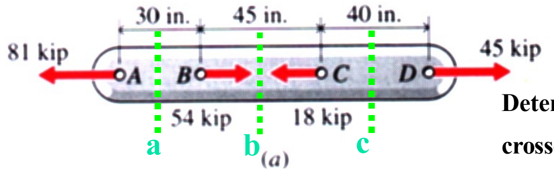
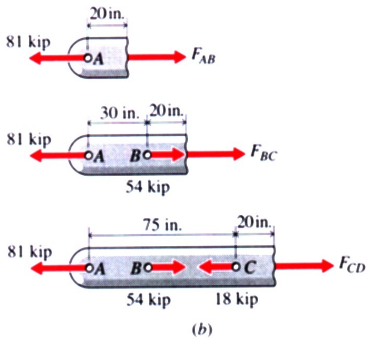
Example Problem 2-1(a)

$A = 3.00\ \text{in}^2$



Determine normal stress at cross-section a, b, and c=?



$\sum F = F_{AB} - 81 = 0$ $\qquad F_{AB} = 81\,\text{kip (T)}$

$\sum F = F_{BC} + 54 - 81 = 0$ $\qquad F_{BC} = 27\,\text{kip (T)}$

$F_{CD} = ?$

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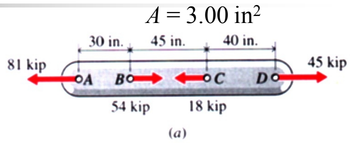
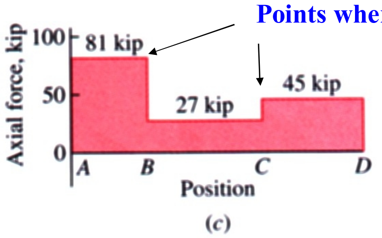
Example Problem 2-1(b)

$\sigma_{CD} = ?$



$$\sigma_{CD} = \frac{F_{CD}}{A} = \frac{+45}{3.00} = +15.00 \text{ ksi} = 15.00 \text{ ksi (T)}$$

- Can you draw the axial-force Diagram?

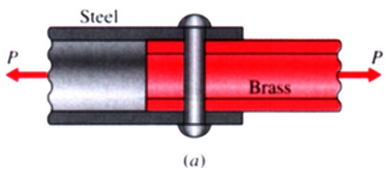


Example Problem 2-3 (a)

$P = 10$ **kips**

$d_s^i = 2.00 \text{ in}$

$t_s = 0.250 \text{ in}$

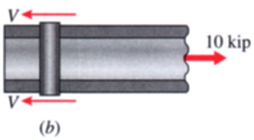


$d_b^o = 2.00 \text{ in}$

$t_b = 0.275 \text{ in}$

$d_p = 0.750 \text{ in}$

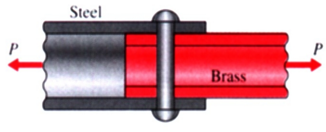
- Determine shearing stress in the pin = ?



$2V = 10 \qquad V = 5 \text{ kips}$

$$\tau = \frac{V}{A} = \frac{5}{\frac{\pi}{4}(0.750)^2} = 11.317 \text{ ksi} \cong 11.32 \text{ ksi}$$

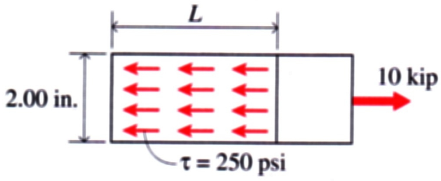
Example Problem 2-3 (b)



(a)

- Replace the pin by a glued joint. Length of glued joint = ?

$\tau(\text{glue}) \le 250 \text{ psi}$



(c)

$$A = \pi d L = \pi(2.00)L = 2.00\pi L \text{ in}^2$$

$$\tau = \frac{V}{A} = \frac{10{,}000}{2.00\pi L} = 250 \text{ psi}$$

$$L = 6.366 \text{ in} \cong 6.37 \text{ in}$$

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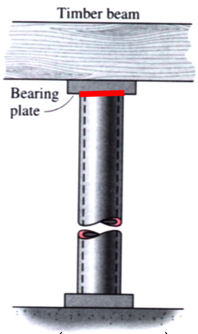
Example Problem 2-4 (a)



$$d_s^o = 150 \text{ mm}$$

$$t_s = 15 \text{ mm}$$

axial load = 150 kN

- Determine average bearing stress between column and bearing plate = ?

$$A = \frac{\pi}{4}\left(d_o^2 - d_i^2\right) = \frac{\pi}{4}\left(150^2 - 120^2\right) = 6362 \text{ mm}^2 = 6362(10^{-6}) \text{ m}^2$$

$$\sigma_b = \frac{F}{A} = \frac{150(10^3)}{6362(10^{-6})} = 23.58(10^6) \text{ N/m}^2 \cong 23.6 \text{ MPa}$$

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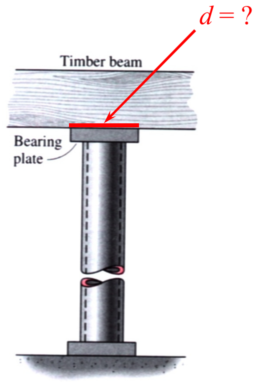
Example Problem 2-4 (b)

- Assuming average bearing stress of bearing plate $\leq 3.25\text{MPa}$

  Determine diameter of bearing plate $\boldsymbol{d} = ?$



$$A = \frac{\pi}{4} d^2$$

$$\sigma_b = \frac{F}{A} = \frac{150(10^3)}{(\pi/4)d^2} = 3.25(10^6)$$

$$d = 242.4(10^{-3})\text{ m} \cong 242\text{ mm}$$

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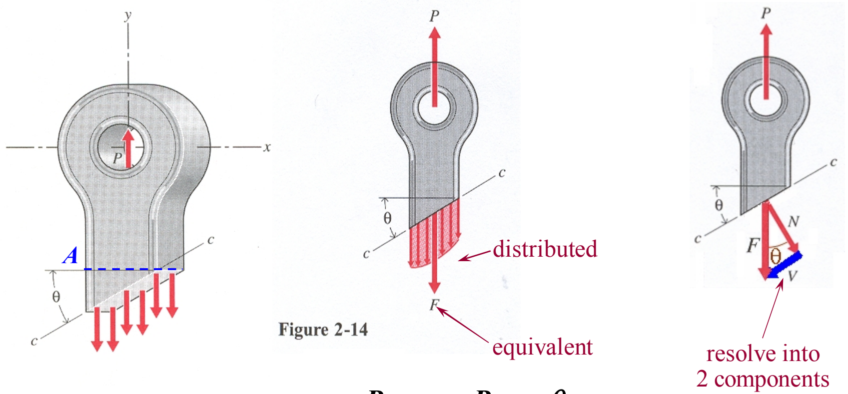
2-6 Stresses on an *Inclined Plane* in an Axially Loaded Member



Average total stress $s_{avg} = \dfrac{\boldsymbol{P}}{\boldsymbol{A}/\cos\boldsymbol{\theta}} = \dfrac{\boldsymbol{P}\cos\boldsymbol{\theta}}{\boldsymbol{A}}$

Not useful for design!

$$|N| = \boldsymbol{P}\cos\theta$$

$$|V| = \boldsymbol{P}\sin\theta$$

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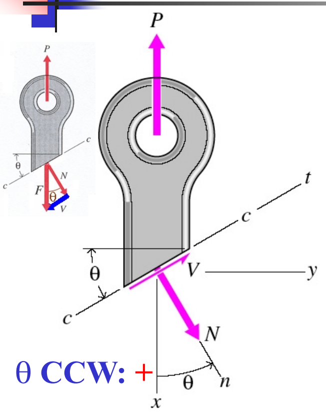
2-6 Stresses on an Inclined Plane in an Axially Loaded Member



θ CCW: +

$N = P\cos\theta$

$V = -P\sin\theta$

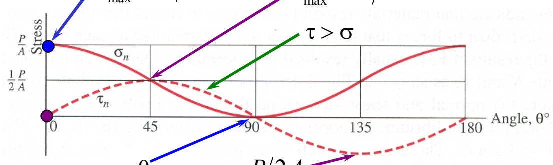
Axially loaded only!

Assumption: stress uniformly distributed

$$\sigma_n = \frac{N}{A_n} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta = \frac{P}{2A}(1+\cos 2\theta) \quad (2\text{-}7)$$

$$\tau_n = \frac{V}{A_n} = -\frac{P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta = -\frac{P}{2A}\sin 2\theta \quad (2\text{-}8)$$

$\sigma_{max} = P/A$ $\tau_{max} = P/2A$ $\tau > \sigma$



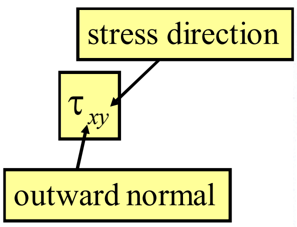
$\sigma_{min} = 0$ $\tau_{min} = -P/2A$

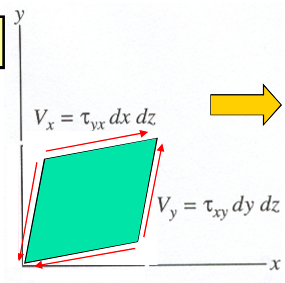
σ and τ vanish at 90°

Remarks for τ_{xy} & τ_{yx}

- The equality of τ_n on orthogonal planes can be obtained as follows:

$$\sum M_z = 0: \ \tau_{yx}(dx\,dz)dy = \tau_{xy}(dy\,dz)dx$$





$$\tau_{yx} = \tau_{xy}$$

Shear stresses tend to "skew" the element diagonally.

- $\tau_{yx} = \tau_{xy}$ is true even when there are normal stresses.

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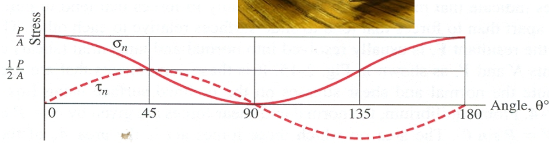
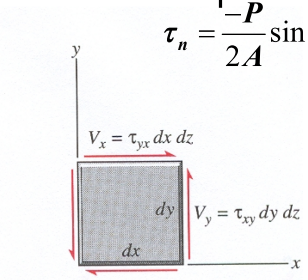
Remarks for τ_{xy} & τ_{yx}

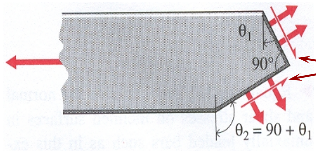
- Failure plane under axial tension loading:
  - brittle material: $\theta \sim 0^{o}$
  - ductile material: $\theta \sim 45^{o}$

- $\tau(\theta) = -\tau(\theta + 90^{\circ})$

$$\tau_n = \frac{-P}{2A}\sin 2\theta$$







Equal magnitude
Change sense (sign)

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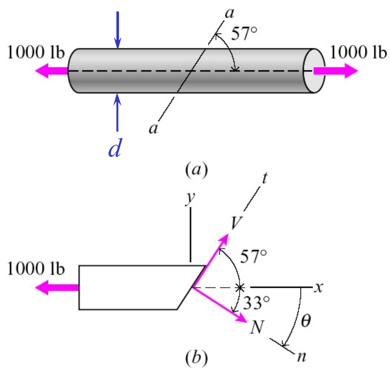
Example Problem 2-6

Given: $d = 1.25$ in.



- Find: σ_n, τ_n

$$\sigma_n = \frac{N}{A_n} = \frac{P}{2A}(1 + \cos 2\theta) \qquad (2\text{-}7)$$

$$= \frac{1000}{2 \cdot \pi \cdot 1.25^2/4}\{1 + \cos[2 \cdot (-33^{\circ})]\}$$

$$= 573 \text{ psi}$$

$$\tau_{nt} = \frac{-P}{2A}\sin 2\theta \qquad (2\text{-}8)$$

$$= \frac{-1000}{2 \cdot \pi \cdot 1.25^2/4}\sin[2 \cdot (-33^{\circ})]$$

$$= 372 \text{ psi}$$

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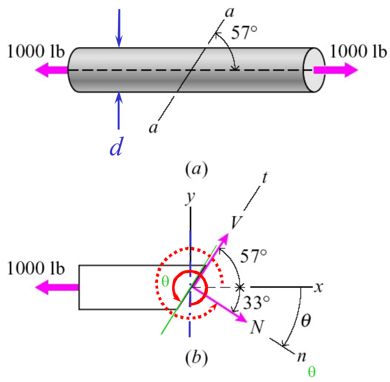
Example Problem 2-6

For $d = 1.25$ in.



Alternatively,

$$\sigma_n = \frac{P}{2A}(1+\cos 2\theta) \qquad \text{(2-7)}$$

$$= \frac{1000}{2\cdot\pi\cdot 1.25^2/4}[1+\cos(2\cdot 327°)]$$

$$= 573 \text{ psi}$$

$$\tau_{nt} = \frac{-P}{2A}\sin 2\theta \qquad \text{(2-8)}$$

$$= \frac{-1000}{2\cdot\pi\cdot 1.25^2/4}\sin(2\cdot 327°)$$

$$= 372 \text{ psi}$$

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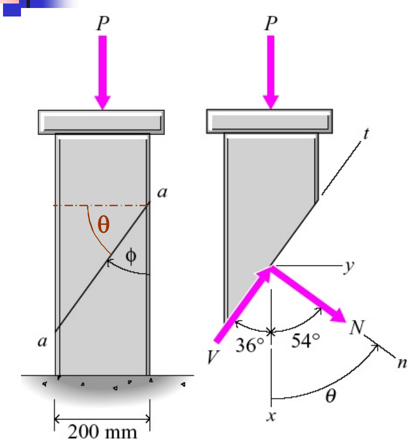
Example Problem 2-7



Given: rectangular block with cross section: 200 × 100 mm, $\phi = 36°$,

σ_n = 12.00 MPa (C) at section *a-a*

- Find:
  - P = ?
  - On plane *a-a*, τ_n = ?
  - Maximum normal and shearing stresses in the block = ?

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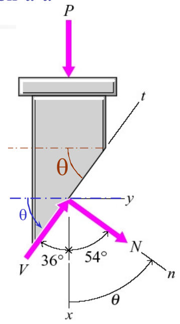
Example Problem 2-7

$\phi = 36^o$,
$\sigma_n = 12.00$ MPa (C) at section a-a



$$A = 200(100) = 20{,}000\ \text{mm}^2 = 0.0200\ \text{m}^2$$

$$A_n = A/\cos\theta = 0.020/\cos 54° = 0.03403\ \text{m}^2$$

$$N = \sigma_n A_n = 12(10^6)(0.03403) = 408.4\ \text{kN} \quad \text{(C)}$$

$$\sum F_n = -408.4 + P\cos 54^\circ = 0$$

$$P = 694.8\ \text{kN} \cong 695\ \text{kN}\ \text{(C)}$$

$$\tau_n = \frac{-P}{2A}\sin 2\theta = \frac{694.8(10^3)}{2(0.0200)}\sin 108° = 16.520(10^6)\ N/m^2 = 16.52\ MPa \quad \text{(eq.2-8)}$$

$$\sigma_{\max} = ?$$

$$\tau_{\max} = ?$$

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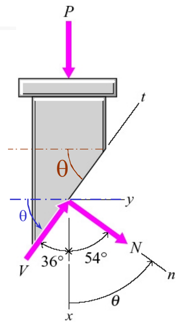
Example Problem 2-7

$$\sigma_{\max} = ?\ \tau_{\max} = ?$$



$$\sigma_{\max} = \frac{P}{A} = \frac{694.8(10^3)}{0.0200} = 34.74(10^6)\ \text{N/m}^2 \cong 34.7\ \text{MPa (C)} \quad \text{(eq.2-9)} \quad \text{at } \theta = 0°$$

$$\tau_{\max} = \frac{P}{2A} = \frac{694.8(10^3)}{2(0.0200)} = 17.35(10^6)\ \text{N/m}^2 \cong 17.35\ \text{MPa} \quad \text{(eq.2-10) at } \theta = 45°$$

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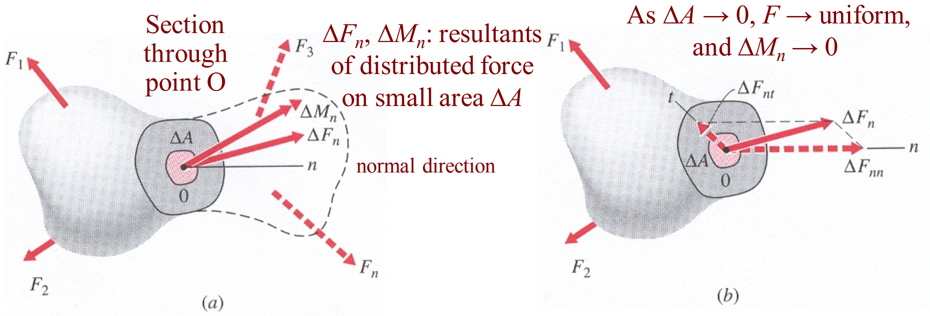
2-7 Stress at a General Point in an Arbitrarily Loaded Member

Stress distribution should not necessarily be uniform in a cross-section.



$$\Delta M_n \to 0 \text{ as } \Delta A \to 0$$

$$S_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} \quad \text{stress vector}$$

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_{nn}}{\Delta A} \quad \text{normal stress}$$

$$\tau_n = \lim_{\Delta A \to 0} \frac{\Delta F_{nt}}{\Delta A} \quad \text{shear stress}$$

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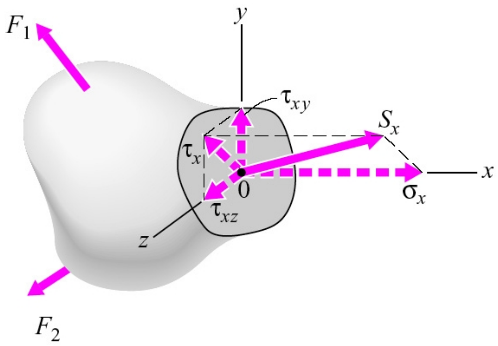
2-7 Stress at a General Point in an Arbitrarily Loaded Member



Resolve τ_x into components τ_{xy} and τ_{xz}

$$S_n = \sigma_n \mathbf{e}_n + \tau_n \mathbf{e}_t$$

$$= \sigma_x \mathbf{e}_x + \tau_{xy} \mathbf{e}_y + \tau_{xz} \mathbf{e}_z$$

Co-plane stress components are vectors.

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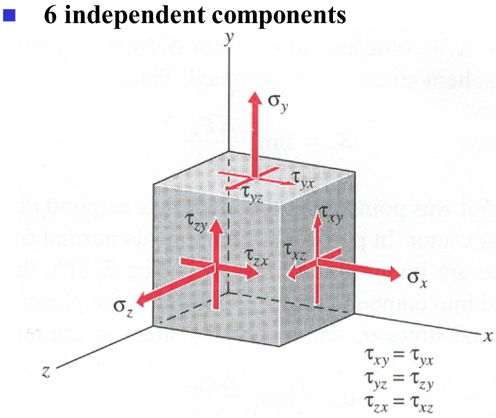
Remarks

- **Resolve S_n on three mutually perpendicular planes**

  ⇒ the state of stress at a point is completely described.
- **Sign convention**
- **6 independent components**



outward normal

τ_{xy}

stress direction

σ_x: normal stress

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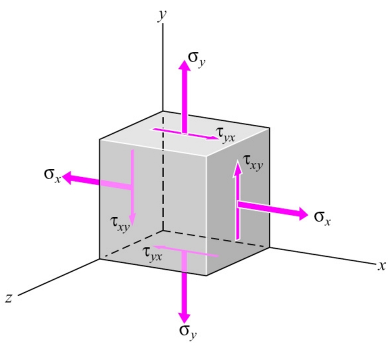
2-8 Two-Dimensional or *Plane Stress*

All forces are confined on a plane, for example, x-y plane

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

which implies

$$\tau_{xz} = \tau_{yz} = 0$$



Plane stress occurs at

- points on the outside surface of a body
- points within thin plates where the z-components of force are zero.

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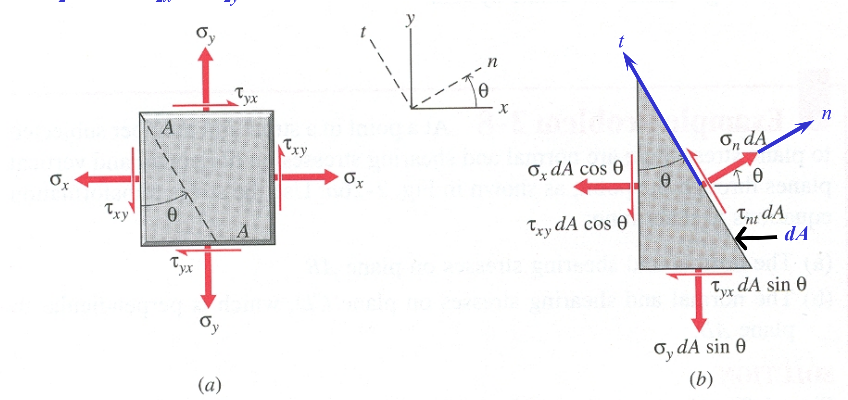
2-9 The Stress Transformation Equations for Plane Stress

$$\sigma_z = 0,\ \tau_{zx} = \tau_{zy} = 0$$



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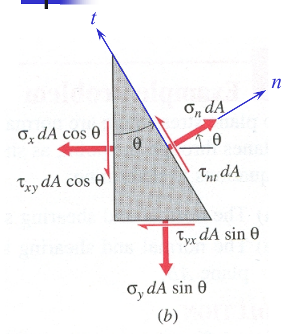
2-9 The Stress Transformation Equations for Plane Stress



$$\overset{+}{\nearrow} \sum F_n = \sigma_n dA - \sigma_x (dA\cos\theta)\cos\theta - \sigma_y (dA\sin\theta)\sin\theta$$

$$-\tau_{yx}(dA\sin\theta)\cos\theta - \tau_{xy}(dA\cos\theta)\sin\theta = 0$$

$$\because\ \tau_{yx} = \tau_{xy}$$

$$\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \quad \text{(2-12a)}$$

$$= \sigma_x\frac{1+\cos 2\theta}{2} + \sigma_y\frac{1-\cos 2\theta}{2} + 2\tau_{xy}\frac{\sin 2\theta}{2}$$

$$= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{(2-12b)}$$

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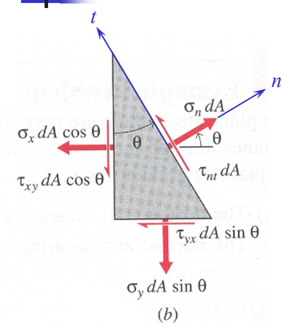
2-9 The Stress Transformation Equations for Plane Stress



$$+\nwarrow \sum F_t = \tau_{nt} dA + \sigma_x (dA\cos\theta)\sin\theta - \sigma_y (dA\sin\theta)\cos\theta$$
$$- \tau_{xy}(dA\cos\theta)\cos\theta + \tau_{yx}(dA\sin\theta)\sin\theta = 0$$

$$\because \quad \tau_{yx} = \tau_{xy}$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad \text{(2-13b)}$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \text{(2-12b)}$$

For Plane stress

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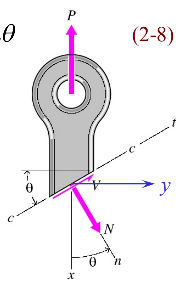
Comparison

2-6 Stresses on an Inclined Plane in an Axially Loaded Member

$$\sigma_n = \frac{P}{2A}(1+\cos 2\theta) \qquad \text{(2-7)}$$

$$\frac{P}{A} = \sigma_x; \quad \sigma_y = 0; \quad \tau_{xy} = 0$$

$$\tau_n = \frac{-P}{2A}\sin 2\theta \qquad \text{(2-8)}$$



2-8 Plane Stresses

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_n = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos 2\theta$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{nt} = -\frac{\sigma_x}{2}\sin 2\theta$$

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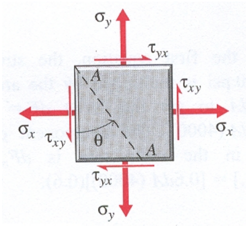
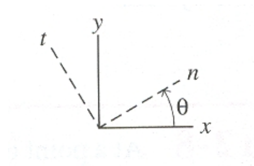
Remarks

Sign conventions:

- positive stresses τ_{ij} or $\tau_{(-i)(-j)}$, and $+\theta$ CCW from $+x$-axis





- (n, t, z) axes have the same order as the (x,y,z) axes. Both are right-hand coordinate system.

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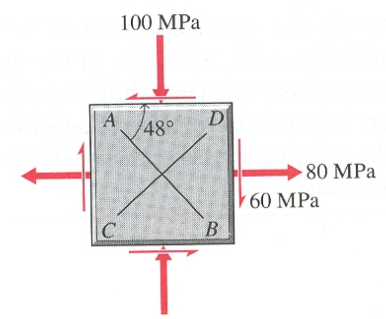
Example Problem 2-8 (I)



- Given: $\sigma_x = +80$ MPa

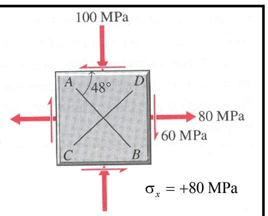
  $\sigma_y = -100$ MPa

  $\tau_{xy} = -60$ MPa

- Find: σ_n and τ_{nt} on plane AB = ?

  σ_n and τ_{nt} on plane CD = ?

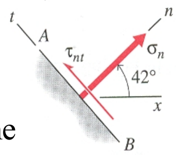
Example Problem 2-8 (II)

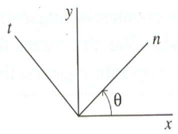


$\sigma_x = +80$ MPa

$\sigma_y = -100$ MPa

$\tau_{xy} = -60$ MPa





$\theta = +42°$

On plane *AB*

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= 80\cos^2(+42°) + (-100)\sin^2(+42°) + 2(-60)\sin(+42°)\cos(+42°)$$

$$= -60.26 \text{ MPa} \cong 60.3 \text{ MPa (C)}$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -[80 - (-100)]\sin(+42°)\cos(+42°) + (-60)(\cos^2(+42°) - \sin^2(+42°))$$

$$= -95.78 \text{ MPa} \cong -95.8 \text{ MPa}$$

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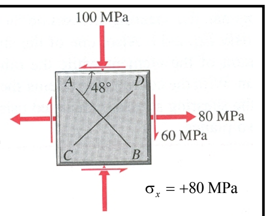
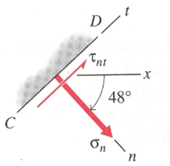
Example Problem 2-8 (III)

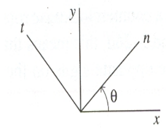


$\sigma_x = +80$ MPa

$\sigma_y = -100$ MPa

$\tau_{xy} = -60$ MPa





$\theta = -48°$

On plane *CD*

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= 80\cos^2(-48°) + (-100)\sin^2(-48°) + 2(-60)\sin(-48°)\cos(-48°)$$

$$= +40.26 \text{ MPa} \cong 40.3 \text{ MPa (T)}$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -[80 - (-100)]\sin(-48°)\cos(-48°) + (-60)(\cos^2(-48°) - \sin^2(-48°))$$

$$= +95.78 \text{ MPa} \cong +95.8 \text{ MPa}$$

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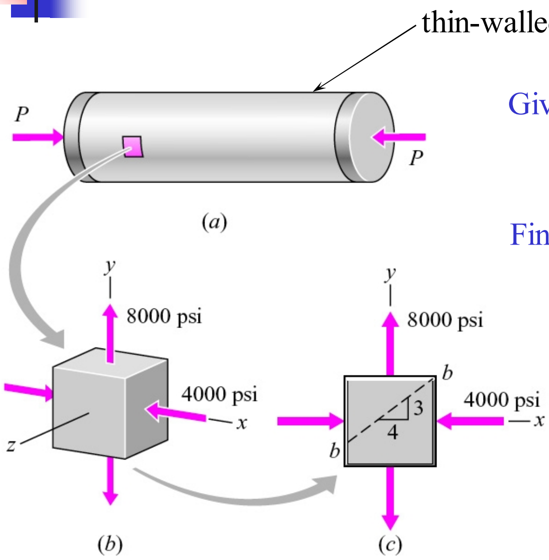
Example Problem 2-9



Given: $\sigma_x = -4000$ psi,
$\sigma_y = 8000$ psi

Find : stresses on plane *b-b*

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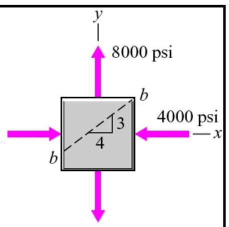
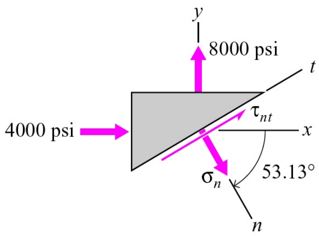
Example Problem 2-9



From Eqs. (2-12a) and (2-13a),



$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

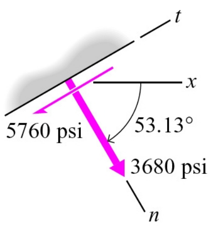
$$= -4000 \cdot \cos^2(-53.13°) + 8000 \cdot \sin^2(-53.13°) + 0$$

$$= 3680 \text{ psi (T)}$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -(-4000-800)\cdot\sin(-53.13°)\cdot\cos(-53.13°) + 0$$

$$= -5760 \text{psi (C)}$$



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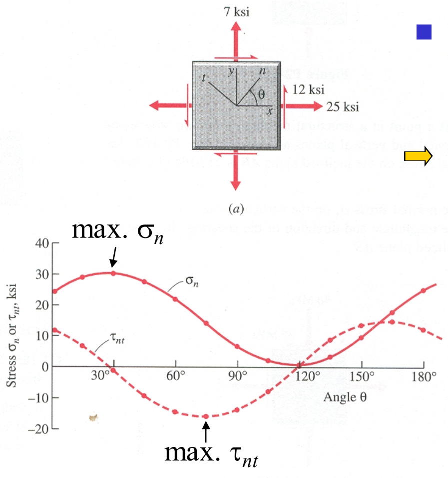
2-10 Principal Stresses and Maximum Shearing Stress – Plane Stress



- Max. σ_n and τ_{nt} (magnitude) can be obtained by plotting curves.

⇨ It's time consuming and inefficient

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Principal Stresses (I)

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

maximize σ_n

⇨ $$\frac{d\sigma_n}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

⇨ $$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$ ⇨ θ_{p1} , θ_{p2} **($\theta_{p1} = \theta_{p2} + 90^o$)**

principal directions (principal planes)

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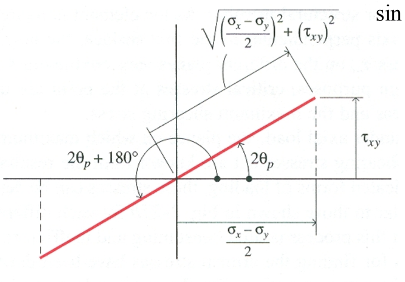
Principal Stresses (II)

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



$$\sin 2\theta_p = \pm\frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}} \qquad \cos 2\theta_p = \pm\frac{\frac{\sigma_x-\sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}$$

substitute into $\tau_{nt} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$

⇨ $\tau_{nt}(\theta_p) = 0$!!

(Principal planes: **Planes free of shear stress**)

substitute into $\sigma_n = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

⇨ $$\sigma_{p1,p2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

Two principal stresses

$$\sigma_{p1} + \sigma_{p2} = \sigma_x + \sigma_y$$

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Maximum Shearing Stress (I)

$$\tau_{nt} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

maximize τ_{nt}

⇨ $$\frac{d\tau_{nt}}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

⇨ $$\tan 2\theta_\tau = -\frac{\sigma_x-\sigma_y}{2\tau_{xy}}$$

⇨ $\theta_{\tau 1}$, $\theta_{\tau 2}$

($\theta_{\tau 1}$ and $\theta_{\tau 2}$ are 90° apart)

$$\tan 2\theta_\tau \times \tan 2\theta_p = -\frac{\sigma_x-\sigma_y}{2\tau_{xy}} \times \frac{2\tau_{xy}}{\sigma_x-\sigma_y} = -1$$

(θ_p and θ_τ are 45° apart)

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Maximum Shearing Stress (II)

$$\sin 2\theta_\tau = \pm \frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}, \qquad \cos 2\theta_\tau = \pm \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

($\theta_{\tau 1}$ and $\theta_{\tau 2}$ are 90° apart)

$$\tau(\theta) = -\tau(\theta + 90°)$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Two maximum shearing stresses with opposite sign.

$$\left(\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\right)$$

$$\Rightarrow \tau_{max} = (\tau_p) = \frac{\sigma_{p1} - \sigma_{p2}}{2}$$

τ_{max} (or τ_p): maximum in-plane shearing stress

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Remarks (I)

$$\frac{d\sigma_n}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta \quad (=0 \text{ for } \sigma_{max})$$

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

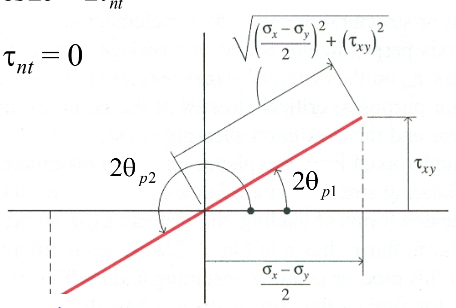
- $\frac{d\sigma_n}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 2\tau_{nt}$

  ⇨ On principal planes, $\tau_{nt} = 0$

  (or $\tau_{nt}(\theta_p) = 0$!!)

- $\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$

  ⇨ $\theta_{p2} = \theta_{p1} + 90°$



Principal planes are orthogonal.

- For plane stress problems, $\sigma_{p3} = \sigma_z = 0$

- $\sigma_{p1} + \sigma_{p2} = \sigma_x + \sigma_y$ ← $\left(\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\right)$

  The sum of two normal stresses on two orthogonal planes are **invariant.**

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Remarks (II)

- $\tan 2\theta_p > 0$ ⇨ $0° < \theta_p < 45°$ 

along z–axis

  $\tan 2\theta_p < 0$ ⇨ $-45° < \theta_p < 0°$ 

along z–axis

- Numerically greater σ_p will act on the plane that makes an angle of 45° or less with the plane of the numerically larger of σ_x and σ_y

- $\tan 2\theta_p = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$ $\qquad \tan 2\theta_\tau = -\dfrac{\sigma_x - \sigma_y}{2\tau_{xy}}$ (Reciprocal relationship)

  ⇨ $\theta_\tau = \theta_p + 45°$ $\quad$ ($\sigma_{p1,p2} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$)

- On θ_τ , $\quad \tau_{max} = \dfrac{\sigma_{p1} - \sigma_{p2}}{2}$, $\quad \sigma_n = \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_{p1} + \sigma_{p2}}{2}$

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Remarks (III)

- 3D case:
  - There are 3 orthogonal planes on which $\tau_{nt} = 0$

    ⇨ principal planes

  - $\tau_{max} = \dfrac{\sigma_{max} - \sigma_{min}}{2}$

    acts on planes that bisect the angles between the planes of σ_{max} and σ_{min} (see next page)

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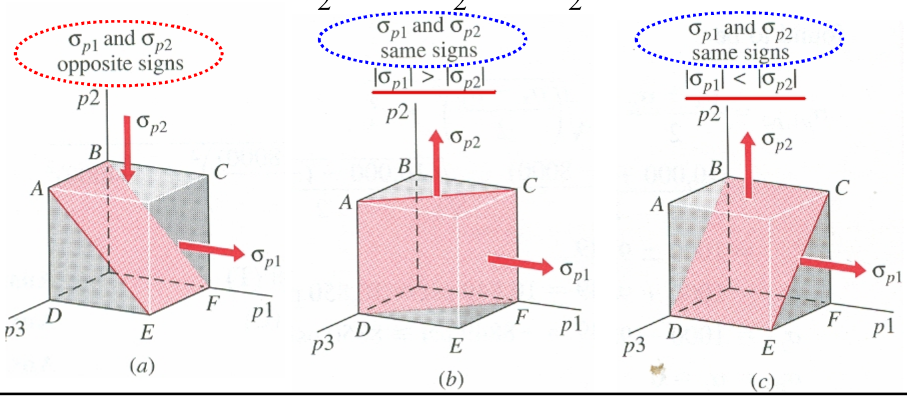
Remarks (IV)

- For **plane stress**:

max. σ_n : σ_{p1} , σ_{p2} , 0 ← σ_{p3}

max. τ_{nt} : $\dfrac{\sigma_{p1}-\sigma_{p2}}{2}$, $\dfrac{\sigma_{p1}-0}{2}$, $\dfrac{\sigma_{p2}-0}{2}$



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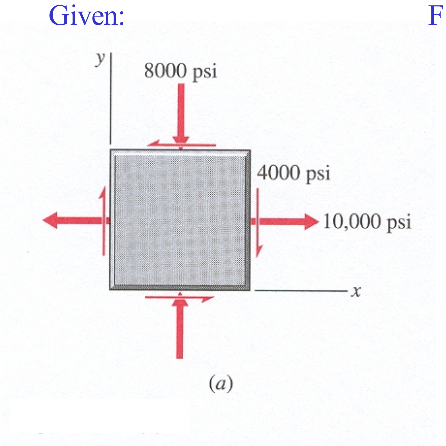
p. 90

Example Problem 2-11

Given:



Find:

- σ_p , τ_{max} =?
- θ_p , θ_τ = ?
- Show the stresses on a sketch.

σ_x = + 10,000 psi

σ_y = – 8000 psi

τ_{xy} = – 4000 psi

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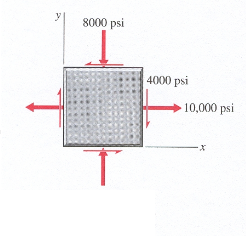
p. 90



Example Problem 2-11

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{(2-15)}$$

$$= \frac{10000 + (-8000)}{2} \pm \sqrt{\left(\frac{10000 - (-8000)}{2}\right)^2 + (-4000)^2}$$

$$= 1000 \pm 9849$$

$$\sigma_{p1} = 1000 + 9849 = 10{,}849 \text{ psi} \cong 10{,}850 \text{ psi (T)} \quad \#$$

$$\sigma_{p2} = 1000 - 9849 = -8849 \text{ psi} \cong 8850 \text{ psi (C)} \quad \#$$

$$\sigma_{p3} = \sigma_z = 0 \quad \# \qquad\qquad \sigma_{p1} \bullet \sigma_{p2} < 0$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{10{,}849 - (-8849)}{2} = 9849 \text{ psi} \cong 9850 \text{ psi} \quad \#$$

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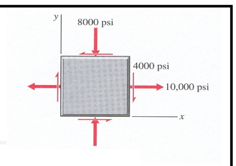


Example Problem 2-11

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-4000)}{10000 - (-8000)} = -0.4444$$

$$2\theta_p = -23.96°, \quad +156.04°$$

$$\theta_p = -11.98°, \quad +78.02° \quad \#$$

- $\theta_p = -11.98°$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

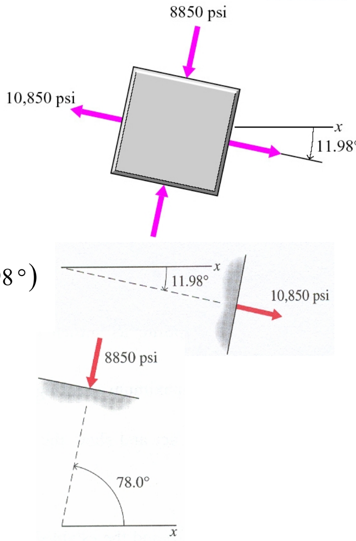
$$= 10000 \cos^2(-11.98°) + (-8000)\sin^2(-11.98°) + 2(-4000)\sin(-11.98°)\cos(-11.98°)$$

$$= \sigma_{p1} = 10{,}849 \text{ psi} \cong 10{,}850 \text{ psi (T)}$$

- $\theta_p = +78.02°$

$$\sigma_n = 10000\cos^2(78.02°) + (-8000)\sin^2(78.02°) + 2(-4000)\sin(78.02°)\cos(78.02°)$$

$$= \sigma_{p2} = -8849 \text{ psi} \cong -8850 \text{ psi (C)}$$



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Example Problem 2-11

the max. in-plane shear stress

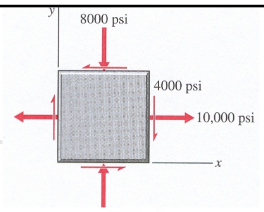
- $\theta_\tau = -11.98° + 45° = +33.02°$

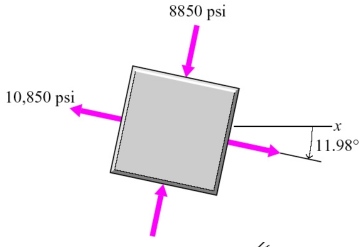
$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

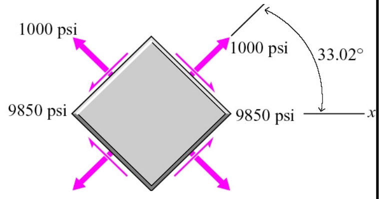
$$\sigma_n = +999.6\ psi \cong 1000\ psi\ (T)$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= -(10000 - (-8000))\sin 33.02° \cos 33.02°$$
$$+ (-4000)(\cos^2 33.02° - \sin^2 33.02°)$$
$$= -9849\ \text{psi} \cong -9850\ \text{psi}$$

(On θ_τ , $\sigma_n = \frac{\sigma_x + \sigma_y}{2}; \tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}$)







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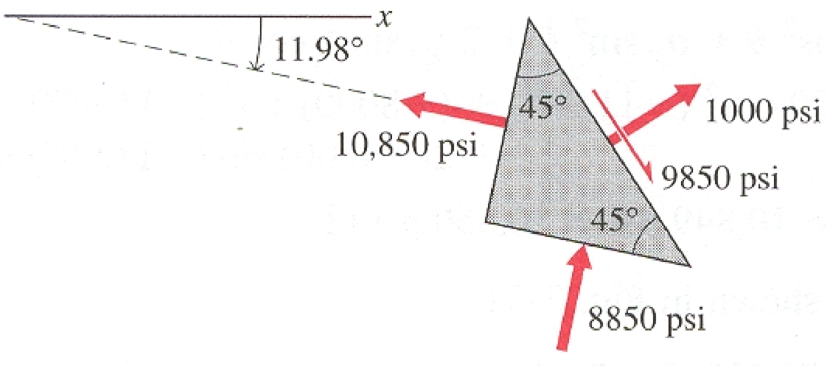
Example Problem 2-11



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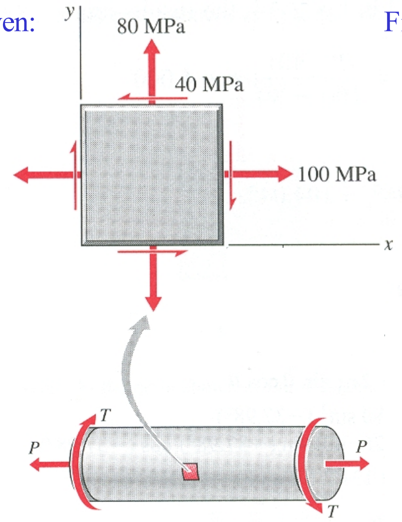
Example Problem 2-12

Given:



(a)

Find:

- σ_p , τ_{max} =?
- θ_p , θ_τ = ?
- Show the stresses on a sketch.

$$\sigma_x = +100 \text{ MPa}$$
$$\sigma_y = +80 \text{ MPa}$$
$$\tau_{xy} = -40 \text{ MPa}$$

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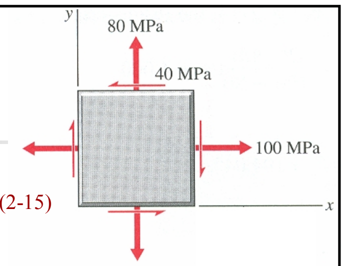
Example Problem 2-12



$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{(2-15)}$$

$$= \frac{100+80}{2} \pm \sqrt{\left(\frac{100-80}{2}\right)^2 + (-40)^2}$$

$$= 90 \pm 41.23$$

$$\sigma_{p1} = 90 + 41.23 = +131.23 \text{ MPa} \cong 131.2 \text{ MPa (T)}$$

$\sigma_{p1} \bullet \sigma_{p2} > 0$

$$\sigma_{p2} = 90 - 41.23 = +48.77 \text{ MPa} \cong 48.8 \text{ MPa (T)}$$

$$\sigma_{p3} = \sigma_z = 0$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{131.23 - 0}{2} = 65.61 \text{ MPa} \cong 65.6 \text{ MPa} \quad \text{(2-19)}$$

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Example Problem 2-12

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-40)}{100-80} = -0.4000 \qquad \text{(2-14)}$$

$$2\theta_p = -75.96°, \quad +104.04°$$

$$\theta_p = -37.98°, \quad +52.02°$$

- $\theta_p = -37.98°$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \qquad \text{(2-12a)}$$

$$= 100\cos^2(-37.98°) + 80\sin^2(-37.98°) + 2(-40)\sin(-37.98°)\cos(-37.98°)$$

$$= \sigma_{p1} = +131.23 \text{ MPa} \cong 131.2 \text{ MPa (T)}$$

- $\theta_p = +52.02°$

$$\sigma_n = 100\cos^2(+52.02°) + 80\sin^2(+52.02°) + 2(-40)\sin(+52.02°)\cos(+52.02°)$$

$$= \sigma_{p2} = +48.77 \text{ MPa} \cong 48.8 \text{ MPa (T)}$$

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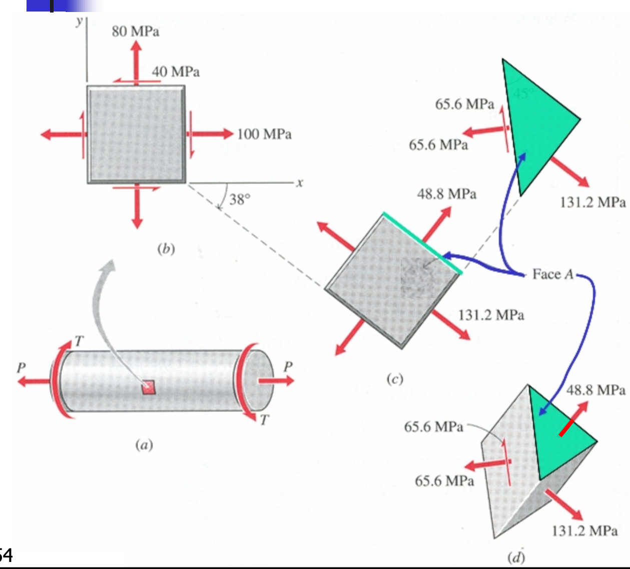
p. 94

Example Problem 2-12



Note:

In *xy*-plane,

$\tau_p = (\sigma_{p1} - \sigma_{p2})/2$

$= (131.2-48.8)/2$

$= 41.2$ MPa

acting on $\theta = 38° + 45°$

$\tau_p < \tau_{max} = 65.6$ MPa

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2-11 Mohr's Circle for Plane Stress

Otto Mohr (German engineer, 1835-1918)

$A = 2\theta_p$

$B = 2\theta$

cos(A-B) = cos(A) *cos*(B) + sin(A) sin(B)

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad = \sigma_{avg} + R\cos\left(2\theta_p - 2\theta\right)$$

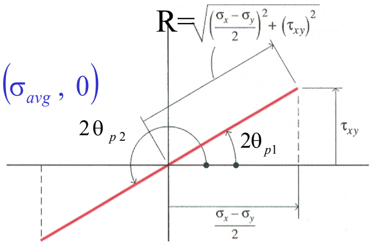
$$\tau_{nt} = -\sin 2\theta \frac{\sigma_x - \sigma_y}{2} + \cos 2\theta\ \tau_{xy} \quad = R\sin\left(2\theta_p - 2\theta\right)$$

square sum sin(-B+A) = sin(-B)*cos*(*A*) + cos(-B)sin(A)

$$\Rightarrow \left(\sigma_n - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{nt}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Circle : center $\left(\sigma_n\ ,\ \tau_{nt}\right) = \left(\frac{\sigma_x + \sigma_y}{2}\ ,\ 0\right) = \left(\sigma_{avg}\ ,\ 0\right)$

radius $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$



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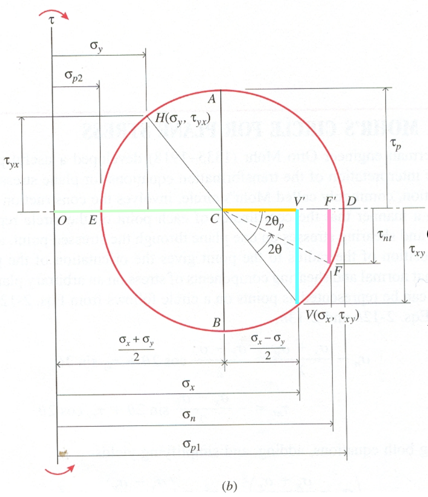
2-11 Mohr's Circle for Plane Stress



$$OF' = OC + CF\cos\left(2\theta_p - 2\theta\right)$$

$= \left(\sigma_x + \sigma_y\right)/2 = \sigma_{avg}$ $\qquad = VV' = \tau_{xy}$

$$OF' = OC + CV\cos 2\theta_p \cos 2\theta + CV\sin 2\theta_p \sin 2\theta$$

$= CV' = \left(\sigma_x - \sigma_y\right)/2$

$$OF' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta = \sigma_n$$

$$= CF\sin\left(2\theta_p - 2\theta\right)$$

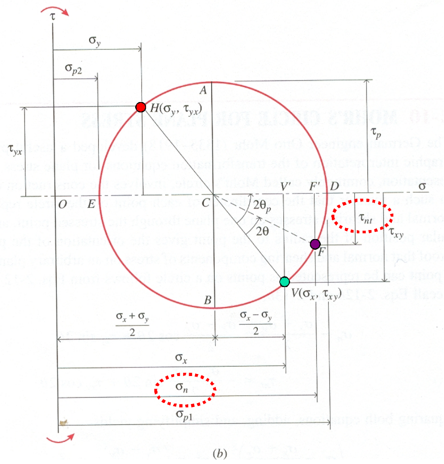
$$= CV\sin 2\theta_p \cos 2\theta - CV\cos 2\theta_p \sin 2\theta$$

$$= V'V\cos 2\theta - CV'\sin 2\theta$$

$$= \tau_{xy}\cos 2\theta - \frac{\sigma_x - \sigma_y}{2}\sin 2\theta = \tau_{nt}$$

p. 100

2-11 Mohr's Circle for Plane Stress



$\sigma_n = \sigma_{avg} + R\cos\left(2\theta_p - 2\theta\right)$

$\tau_{nt} = R\sin\left(2\theta_p - 2\theta\right)$ ⇨ Point F ●

■ $\theta = 0^\circ$

$\sigma_n = \sigma_{avg} + R\cos 2\theta_p = \sigma_x$

$\tau_{nt} = R\sin 2\theta_p = \tau_{xy}$ ⇨ Point V ●

■ $\theta = 90^\circ$

$\sigma_n = \sigma_{avg} - R\cos 2\theta_p = \sigma_y$

$\tau_{nt} = -R\sin 2\theta_p = -\tau_{xy}$ ⇨ Point H ●

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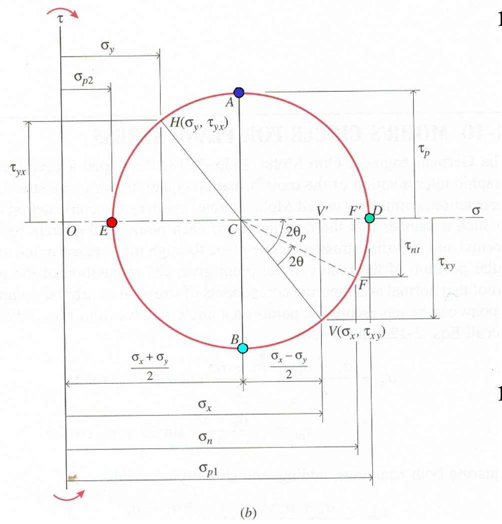
p. 100

2-11 Mohr's Circle for Plane Stress



max. normal stress:

$\sigma_{p1} = \sigma_{avg} + R\cos(2\theta_p - 2\theta_p)$ ●

$= \sigma_{avg} + R$

$\tau_{nt} = -R\sin(2\theta_p - 2\theta_p) = 0$

$\sigma_{p2} = \sigma_{avg} + R\cos(2\theta_p - 2\theta_p - 180°)$ ●

$= \sigma_{avg} - R$

$\tau_{nt} = -R\sin(2\theta_p - 2\theta_p - 180°) = 0$

max. in-plane shearing stress

$\tau_p = R\sin(2\theta_p - 2\theta_p \pm 90°) = \pm R$ ● ●

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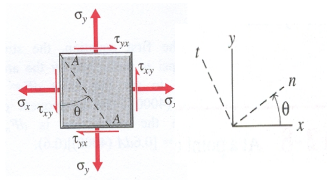
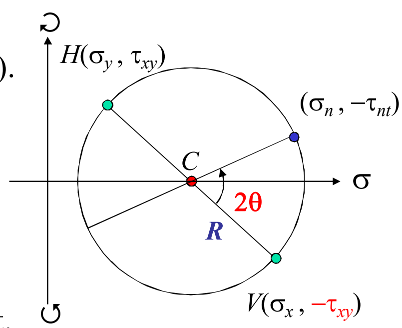
Procedure for Drawing Mohr's Circle

- Choose a set of x-y axes.
- Identify σ_x, σ_y , and τ_{xy} with proper sign.
- Draw a set of σ–τ axes.
- Plot points $V(\sigma_x , -\tau_{xy})$ and $H(\sigma_y , \tau_{xy})$.
- Draw line VH and determine the center C and radius R.
- Draw the circle.
- Use CV as the x-axis ($\theta = 0^o$) or the reference line for angle measurement.





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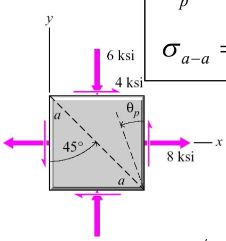
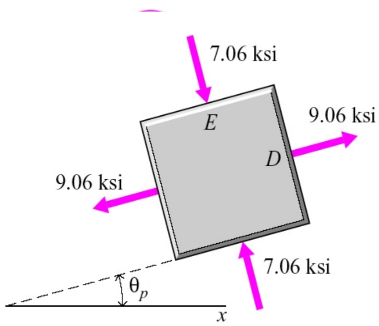
Example Problem 2-13

$\sigma_p = ? \tau_{max} = ? \theta_p = ?$

$\sigma_{a-a} = ? \tau_{a-a} = ?$





$$OC = (\sigma_x + \sigma_y)/2 = (8-6)/2 = 1$$

$$CV = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{7^2 + 4^2} = 8.062\,\text{ksi}$$

$$\sigma_{p1} = OD = OC + CV = 1 + 8.062 = 9.06\,\text{ksi}$$

$$\sigma_{p2} = OE = CE - CV = 1 - 8.062 = -7.06\,\text{ksi}$$

$$\sigma_{p3} = \sigma_z = 0$$

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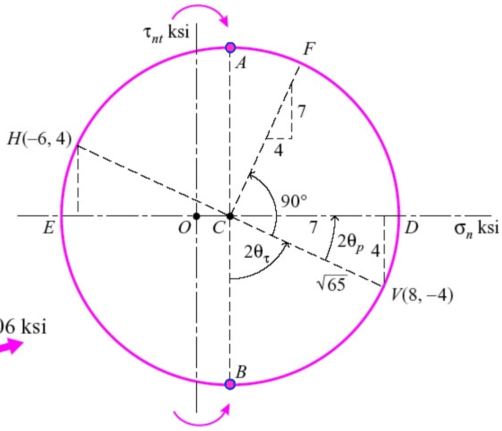
Example Problem 2-13

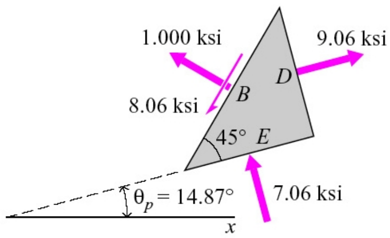
$$\tau_p = \tau_{max} = CA = CB = 8.06\ \text{ksi}$$

$$\sigma_n = OC = 1.000\ \text{psi}$$

$$2\theta_p = \tan^{-1}(4/7) = 29.74°$$

$$\theta_p = 14.87°$$





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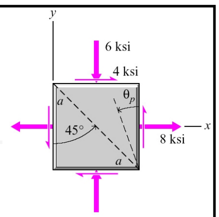
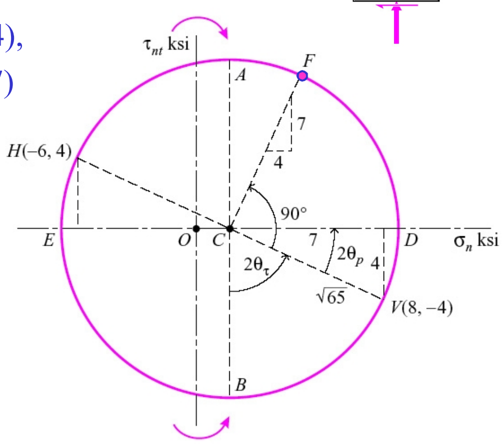
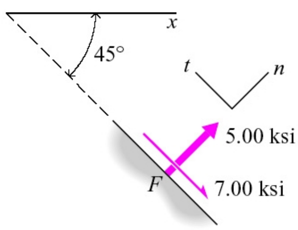
Example Problem 2-13



At plane *a-a*, 45º from *V*(8, –4), 90º CCW from *CV*, i.e., *F*(5, 7)





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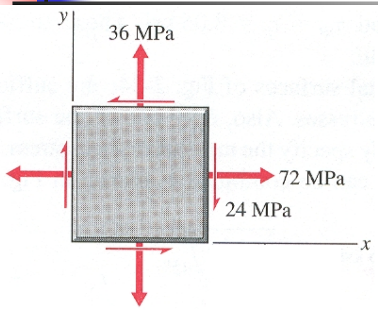
Example Problem 2-14



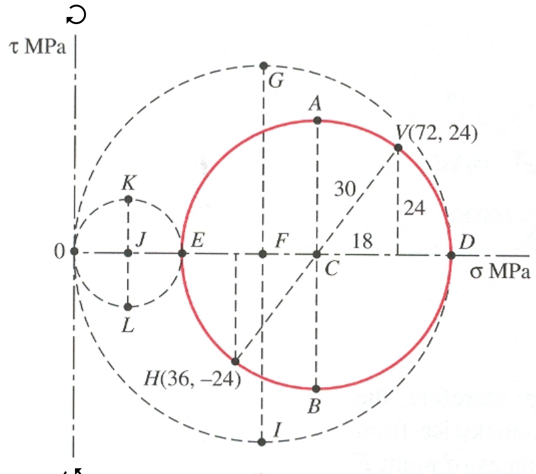
$$\sigma_{p1} = OD = 54 + 30 = 84.0 \text{ MPa (T)}$$

$$\sigma_{p2} = OE = 54 - 30 = 24.0 \text{ MPa (T)}$$

$$\sigma_{p3} = \sigma_z = 0$$

$$\tan 2\theta_p = -24/18 = -1.3333$$

$$\theta_p = -26.57° = 26.57° ↻$$



(b)

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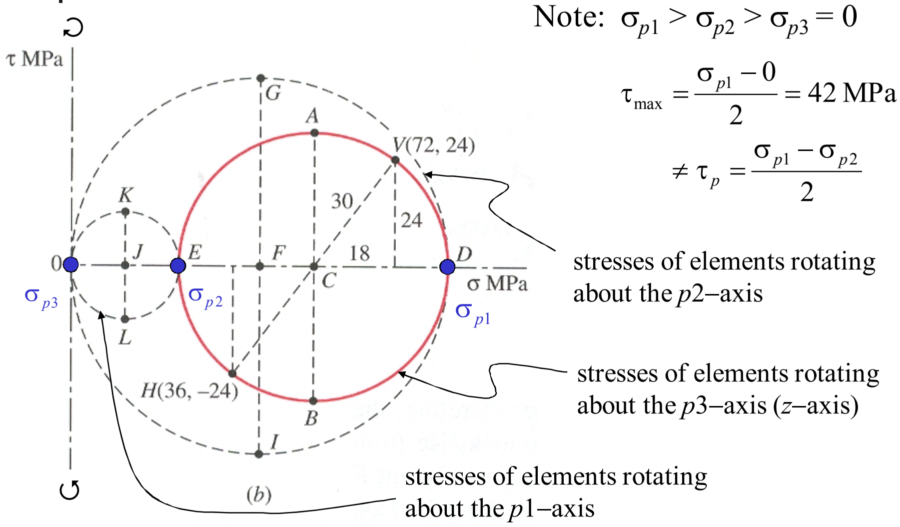
Example Problem 2-14

Note: $\sigma_{p1} > \sigma_{p2} > \sigma_{p3} = 0$

$$\tau_{max} = \frac{\sigma_{p1} - 0}{2} = 42 \text{ MPa}$$

$$\neq \tau_p = \frac{\sigma_{p1} - \sigma_{p2}}{2}$$



(b)

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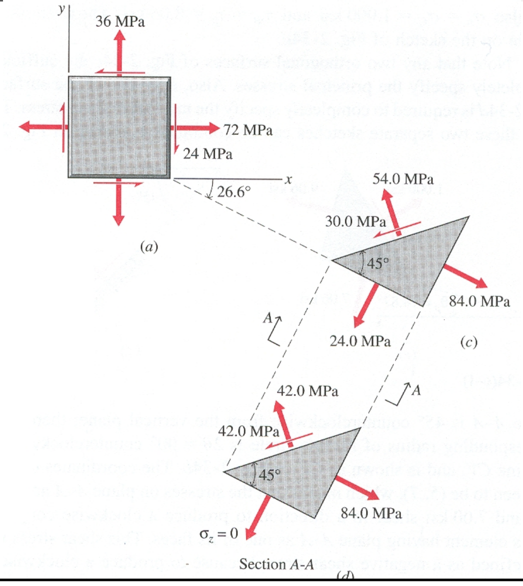
Example Problem 2-14



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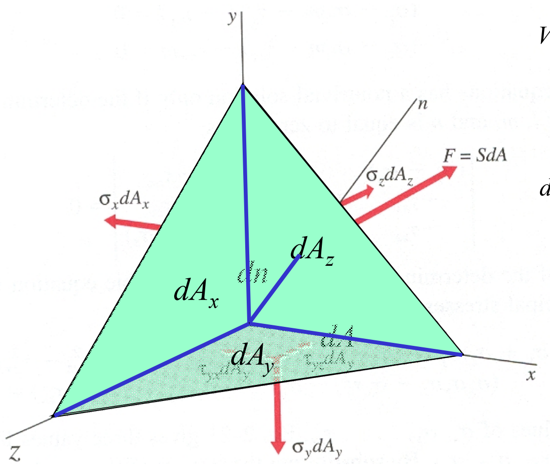
2-12 General State of Stress at a Point



$$V = \frac{1}{3} dn\ dA$$

$$= \frac{1}{3} dx\ dA_x = \frac{1}{3} dy\ dA_y = \frac{1}{3} dz\ dA_z$$

$$dn = dx \cos\theta_x = dy \cos\theta_y = dz \cos\theta_z$$

⇨

$$dA_x = dA \cos\theta_x = dA\ l$$

$$dA_y = dA \cos\theta_y = dA\ m$$

$$dA_z = dA \cos\theta_z = dA\ n$$

l, m, n: direction cosines

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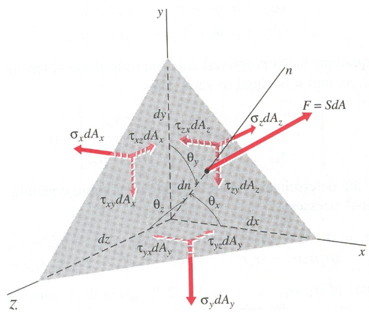
2-12 General State of Stress at a Point

Force equilibrium

$$F_x = S_x dA = \sigma_x dA\, l + \tau_{yx} dA\, m + \tau_{zx} dA\, n$$

$$F_y = S_y dA = \tau_{xy} dA\, l + \sigma_y dA\, m + \tau_{zy} dA\, n$$

$$F_z = S_z dA = \tau_{xz} dA\, l + \tau_{yz} dA\, m + \sigma_z dA\, n$$



$$S_x = \sigma_x l + \tau_{yx} m + \tau_{zx} n$$

$$S_y = \tau_{xy} l + \sigma_y m + \tau_{zy} n$$

$$S_z = \tau_{xz} l + \tau_{yz} m + \sigma_z n$$

$$\sigma_n = S_x l + S_y m + S_z n$$

$$= \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{xy} lm + 2\tau_{yz} mn + 2\tau_{zx} nl$$

$$\tau_{nt} = \pm\sqrt{S^2 - \sigma_n^2}$$

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8 Exercises

2-19, 2-41, 2-61, 2-73,
2-80, 2-93, 2-95, 2-119

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Appendix

Greek alphabet

Upper case	Lower case	Name	Upper case	Lower case	Name
Α	α	alpha	Ν	ν	nu
Β	β	beta	Ξ	ξ	xi /ksi/ or /zai/
Γ	γ	gamma	Ο	ο	omicron
Δ	δ	delta	Π	π	pi
Ε	ε	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σ	sigma
Η	η	eta	Τ	τ	tau
Θ	θ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	ϕ	phi /fai/
Κ	κ	kappa	Χ	χ	chi /kai/
Λ	λ	lambda	Ψ	ψ	psi /sai/
Μ	μ	mu	Ω	ω	omega