Mechanics of Materials

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Chapter 7

Flexural (彎曲) Loading: Stresses in Beams

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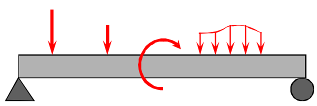
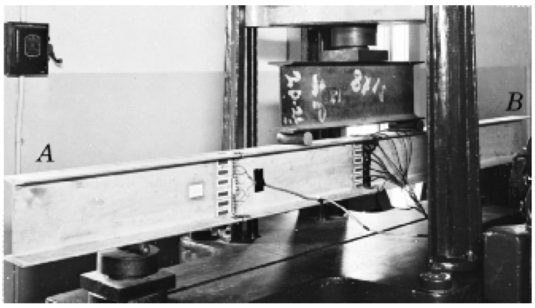
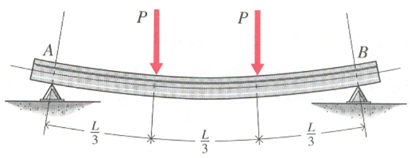
7-1 Introduction

- Beam: a long member subjected to transverse loading



flexural member





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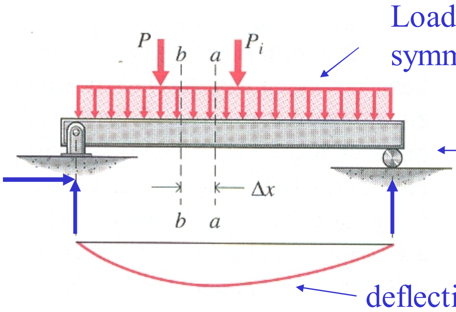
Classification (I)

- Simple beam (statically determinate) 簡支樑

Simply supported beam



Loads act in the plane of symmetry.

Supported by pin, roller, smooth surface

deflection curve

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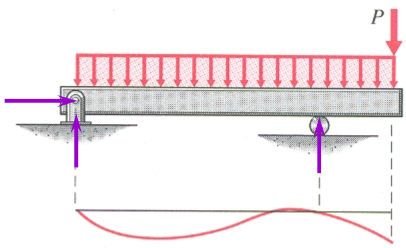
Classification (II)

- Simple beam with overhang (statically determinate)



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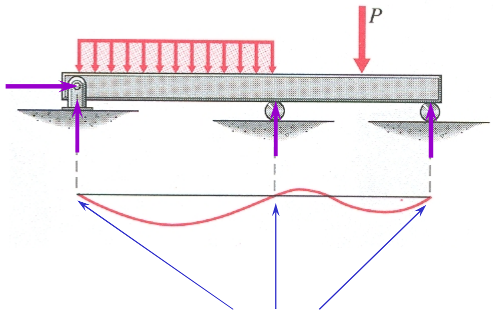
Classification (III)

- Continuous beam (statically indeterminate)



more than two simple supports,
no movements at the supports

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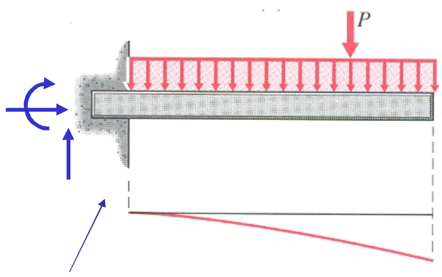
Classification (IV)

- Cantilever beam (statically determinate) 懸臂梁



built-in end
fixed (no rotation occurs)
or restrained (a limited rotation occurs)

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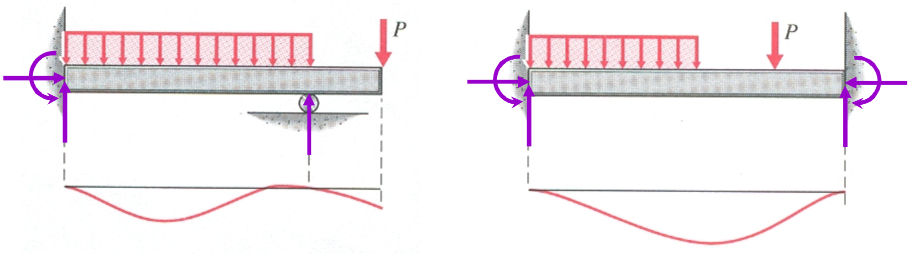
Classification (V)

- Other types (statically indeterminate)



(Fixed end: no movement, no rotation
Restrained end: no movement, limited rotation)

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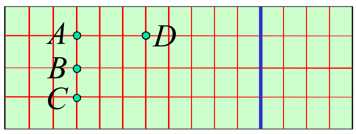
Experiment and Discussion

- Draw grid lines on the rubber



- Bend the rubber and observe its behavior.
  - deformed shape of cross-section
  - elongation of logitudinal fibers
- Can you predict the strains and stresses at point *A*, *B*, *C*, and *D*?

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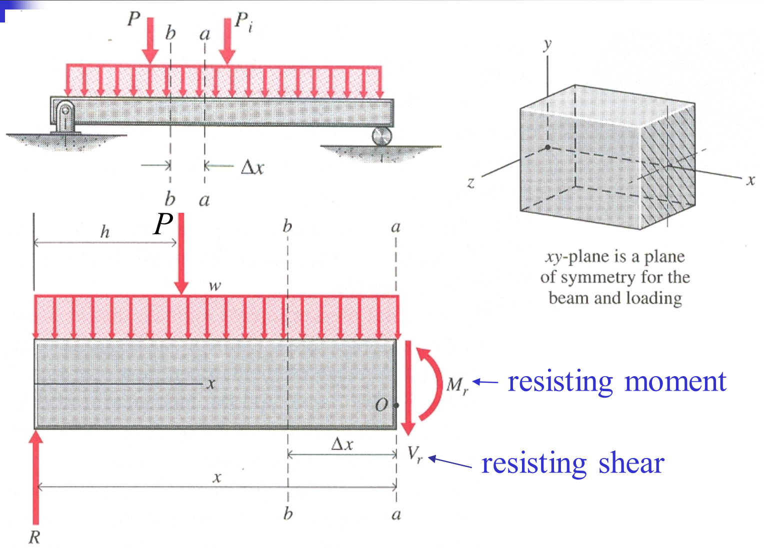
Resisting Moment and Shear



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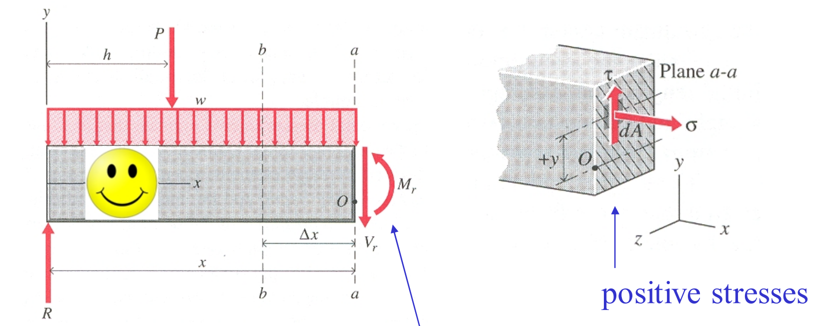
Resisting Moment and Shear

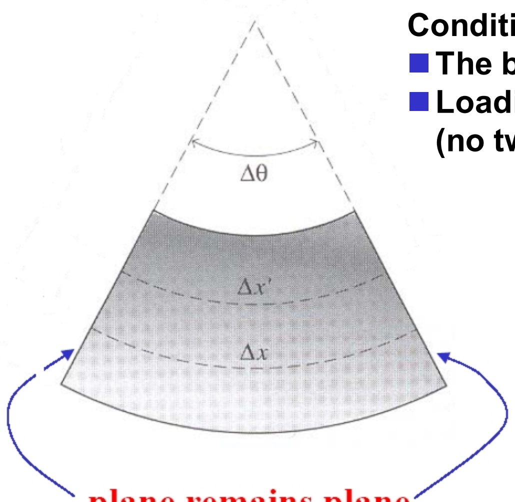


$$V_r = -\int_{\text{area}} \tau \, dA$$

$$M_r = -\int_{\text{area}} y \, \sigma \, dA$$

positive resisting moment M_r
and shear force V_r

7-2 Flexural Strains (I)



Conditions:
- **The beam is bent with couples**
- **Loading in plane of symmetry (no twisting occurs)**

plane remains plane

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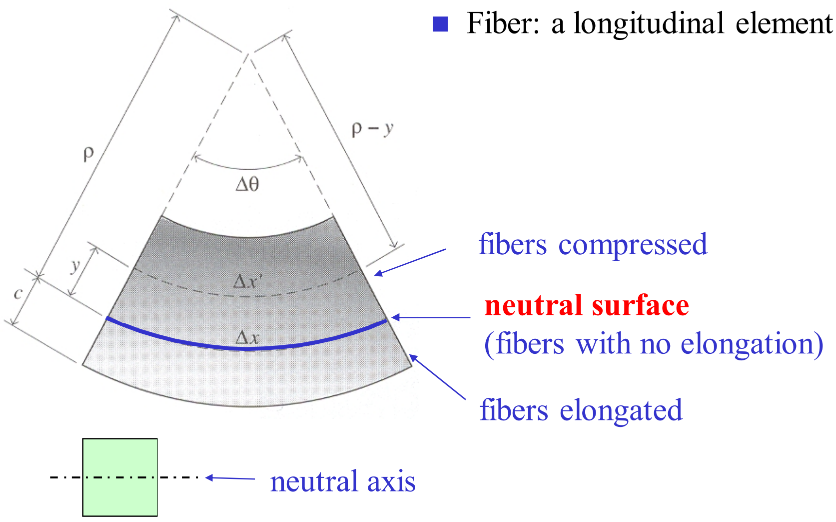
7-2 Flexural Strains (II)

- Fiber: a longitudinal element



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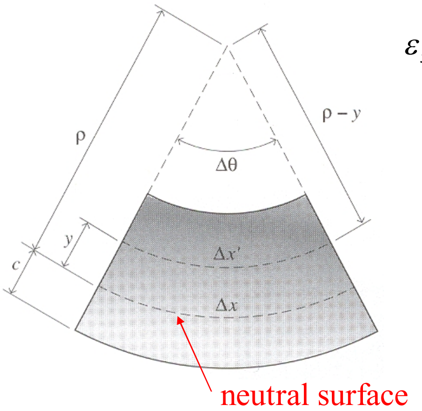
7-2 Flexural Strains (III)

- Assume: All fibers have the same initial length.

  ⇨ **prismatic beam**



$$\varepsilon_x = \frac{\delta}{L} = \frac{L_f - L_i}{L_i}$$

$$= \frac{\Delta x' - \Delta x}{\Delta x} = \frac{(\rho - y)(\Delta\theta) - \rho(\Delta\theta)}{\rho(\Delta\theta)}$$

$$= -\frac{1}{\rho} y$$

valid for elastic or inelastic action

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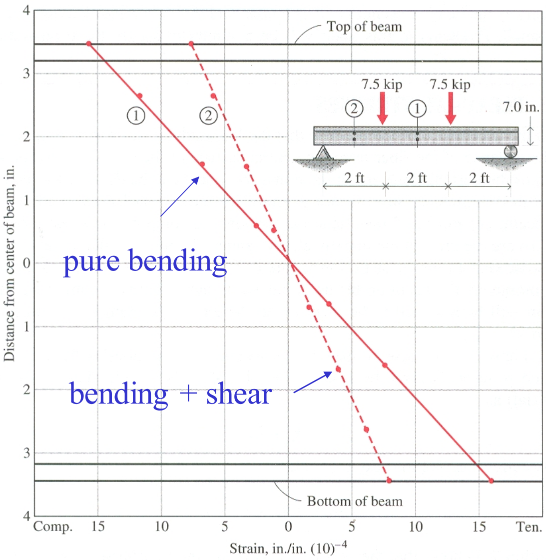
7-2 Flexural Strains (IV)



$\varepsilon_x \propto y$

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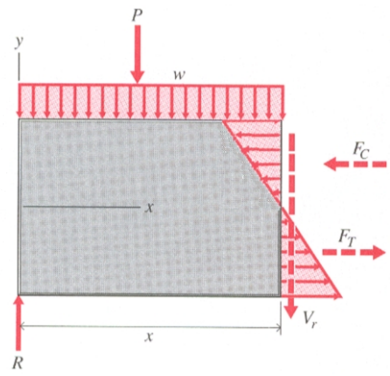
7-3 Flexural Stresses (I)

$$\varepsilon_x = -\frac{1}{\rho} y$$

$$\sigma_x = E\varepsilon_x = -\frac{E}{\rho} y$$

uniaxial stress
valid for pure bending



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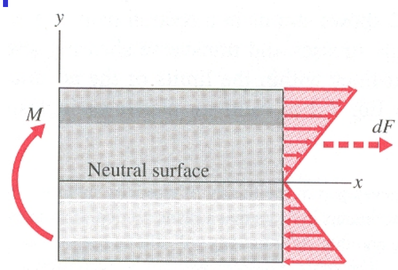
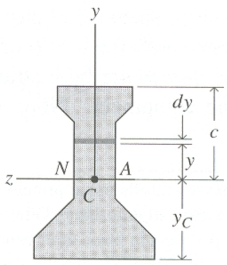
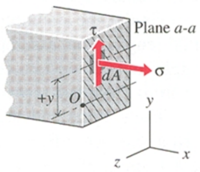
7-3 Flexural Stresses (II)

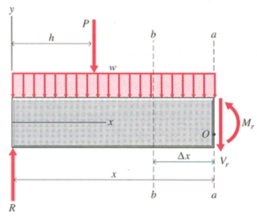






$$M_r = -\int_A y\, dF = -\int_A y\, \sigma_x\, dA$$

Where is the neutral surface?



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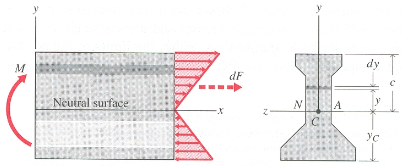
7-3 Flexural Stresses (III)

- Location of Neutral Axis (N.A.)



$$\sum F_x = \int_A \sigma_x \, dA = 0$$

$$\int_A \sigma_x \, dA = \int_A \left(E(-\frac{y}{\rho}) \right) dA = -\frac{E}{\rho} \int_A y \, dA = -\frac{E}{\rho} y_c A = 0$$

y_c: distance from the reference axis (neutral axis) to the centroidal axis *c*-*c* of the cross section

The N.A. passes through the centroid of the cross section for linearly elastic action

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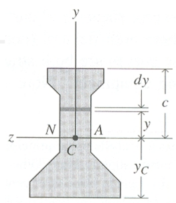
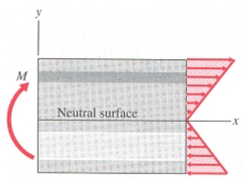
7-3 Flexural Stresses (IV)



$$\sigma_x = -\frac{E}{\rho} y$$

$$\sigma_{max} = -\frac{E}{\rho} c$$ farthest distance from N.A. to the surface

$$\sigma_x = \frac{y}{c} \sigma_{max} = \frac{y}{c} \sigma_c$$



$$M_r = -\int_A y \, \sigma_x \, dA = -\frac{\sigma_c}{c} \int_A y^2 dA$$

I: second moment of area (see Appendix A, A11, Table A-1,)

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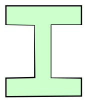
7-4 The Elastic Flexural Formula

$$M_r = -\frac{\sigma_c}{c}\int_A y^2 dA = -\frac{\sigma_c I}{c}$$

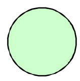
$$\sigma_c = -\frac{M_r c}{I}$$

$$\sigma_x = \frac{y}{c}\sigma_c = -\frac{M_r y}{I} \qquad \boxed{\sigma_x(y) = -\frac{M_r y}{I}}$$

$$\sigma_{\max} = -\frac{M_r c}{I} = -\frac{M_r}{S} \qquad S = \frac{I}{c}$$ **section modulus** of the beam



$S_{\text{I-beam}} > S_{\text{Circle}}$



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Nonsymmetric Sections

- The flexure formula can be applied to nonsymmetric sections if

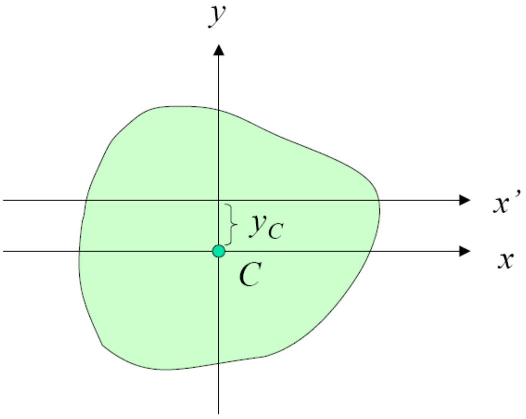
$$\int_A z\,\sigma_x dA = \int_A z\frac{\sigma_c}{c} y dA = \frac{\sigma_c}{c}\int_A zy\, dA = \frac{\sigma_c}{c} I_{yz} = 0$$

- I_{yz}: mixed second moment of the cross sectional area with respect to the centroidal y and z-axes.
- $I_{yz} = 0$ if
  - either y- or z-axis is an axis of symmetry (symm. Section)
  - y- and z-axes are centroidal principal axes (see p. A 17, Appendix A-2-5)

Parallel Axis Theorem

$I_{x'}$: I w.r.t. to x' axis

I_{xC} : I w.r.t. to the centroid



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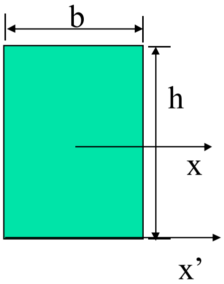
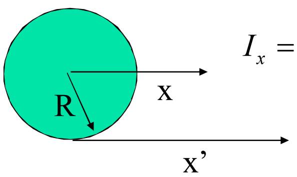
Second Moments of plane areas



$$I_x = \frac{bh^3}{12}$$

$$I_{x'} = \frac{bh^3}{3}$$



$$I_x = \frac{\pi R^4}{4} \qquad I_{x'} = \frac{5\pi R^4}{4}$$

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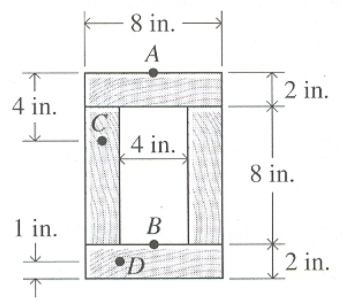
Example Problem 7-1 (I)



- $M_r = -200(10^3)$ in.·lb

Determine

- $\sigma_A = ?$
- $\sigma_B = ?$
- $\sigma_C = ?$
- $\sigma_D = ?$

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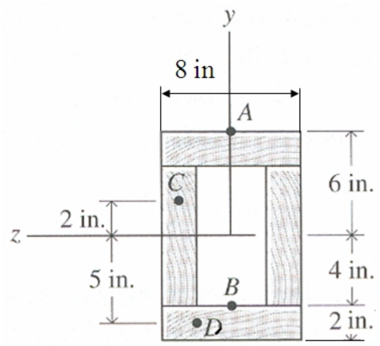
Example Problem 7-1 (II)

- Symmetric section ⇨ centroid = geometric center



$$y_A = +6\text{ in.}$$

$$y_B = -4\text{ in.}$$

$$y_C = +2\text{ in.}$$

$$y_D = -5\text{ in.}$$

$$I = \frac{b_s h_s^3}{12} - \frac{b_h h_h^3}{12} = \frac{8(12)^3}{12} - \frac{4(8)^3}{12} = 981.3\text{ in.}^4$$

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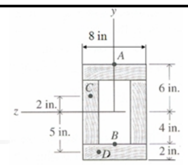
Example Problem 7-1 (III)

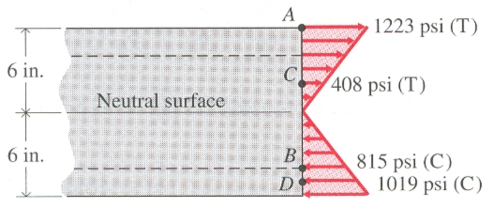


$$\sigma_A = -\frac{M_r y_A}{I} = -\frac{-200(10^3)(+6)}{981.3} = +1222.9\, lb/in.^2 \cong 1223\ psi\ (T)$$

$$\sigma_B = -\frac{M_r y_B}{I} = -\frac{-200(10^3)(-4)}{981.3} = -815.2\, lb/in.^2 \cong 815\ psi\ (C)$$

$$\sigma_C = -\frac{M_r y_C}{I} = -\frac{-200(10^3)(+2)}{981.3} = +407.6\, lb/in.^2 \cong 408\ psi\ (T)$$

$$\sigma_D = -\frac{M_r y_D}{I} = -\frac{-200(10^3)(-5)}{981.3} = -1019.1\, lb/in.^2 \cong 1019\ psi\ (C)$$



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Example Problem 7-1 (IV)

- Alternatively,

$$\sigma_x = -\frac{M_r y}{I} \Rightarrow \frac{\sigma_x}{y} = -\frac{M_r}{I} = \text{constant}$$

$$\Rightarrow \frac{\sigma_A}{y_A} = \frac{\sigma_B}{y_B} = \frac{\sigma_C}{y_C} = \frac{\sigma_D}{y_D}$$

$$\sigma_B = \frac{y_B}{y_A}\sigma_A = \frac{-4}{+6}(1222.9) = -815.3\ \text{lb/in.}^2 \cong 815\ \text{psi (C)}$$

$$\Rightarrow \sigma_C = \frac{y_C}{y_A}\sigma_A = \frac{+2}{+6}(1222.9) = +407.6\ \text{lb/in.}^2 \cong 408\ \text{psi (T)}$$

$$\sigma_D = \frac{y_C}{y_A}\sigma_A = \frac{-5}{+6}(1222.9) = -1019.1\ \text{lb/in.}^2 \cong 1019\ \text{psi (C)}$$

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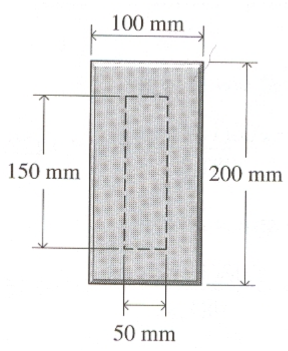
Example Problem 7-2 (I)



- $\sigma_{max} = 15$ MPa

Determine:

- $M_r = ?$
- percentage decrease in $M_r = ?$ if the dotted central protion is removed.

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Example Problem 7-2 (II)

- Symmetric section ⇨ centroid = geometric center

$$\sigma_x = -\frac{M_r y}{I} \qquad M_r = -\frac{\sigma_x I}{y}$$

$$I = \frac{100(200)^3}{12} = 66.67\left(10^6\right)\text{mm}^4 = 66.67\left(10^{-6}\right)\text{m}^4$$

$$|M_r| = \frac{15\left(10^6\right)(66.67)\left(10^{-6}\right)}{100\left(10^{-3}\right)} = 10.00\left(10^3\right)\text{N}\cdot\text{m} = 10.00\ \text{kN}\cdot\text{m}$$

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Example Problem 7-2 (III)

- Hollow section

$$I = \frac{100(200)^3}{12} - \frac{50(150)^3}{12} = 52.60(10^6)\,\text{mm}^4 = 52.60(10^{-6})\,\text{m}^4$$

$$|M_r| = \frac{\sigma_x I}{c} = \frac{15(10^6)(52.60)(10^{-6})}{100(10^{-3})} = 7.89(10^3)\,\text{N}\cdot\text{m} = 7.89\ \text{kN}\cdot\text{m}$$

$$\text{Percent decrease} = D = \frac{10.00 - 7.89}{10.00}100 = 21.1\%$$

Note: A –38% near N.A. M_r – 21%
A –38% from top and bottom M_r – 61%

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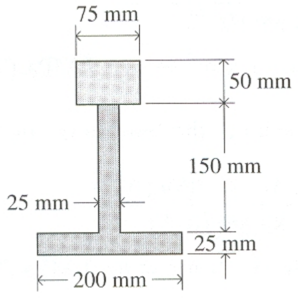
Example Problem 7-3 (I)



- M_r = –75 kN·m

Determine:

- σ_{max} (Tensile) = ?
- σ_{max} (compressive) = ?

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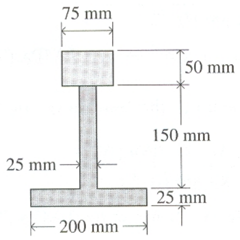
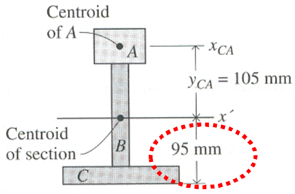
Example Problem 7-3 (II)

- Locate centroid





$$A = 200(25) + 150(25) + 50(75) = 12{,}500 \text{ mm}^2$$

$$M_A(\text{w.r.t. bottom}) = 200(25)(12.5) + 25(150)(100) + 75(50)(200)$$
$$= 1{,}187{,}500 \text{ mm}^3$$

$$y_c = \frac{M_A}{A} = \frac{1{,}187{,}500}{12{,}500} = 95 \text{ mm}$$

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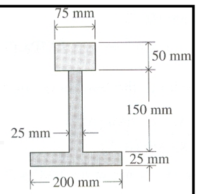
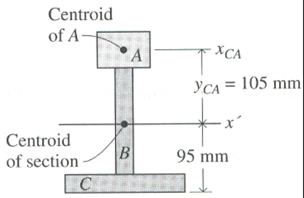
Example Problem 7-3 (III)



- Compute $I_{x'}$ (parallel axis theorem)



$I_{x'} = I_{xC} + Ay_C^2$ $\quad$ $I_{x'}$: I w.r.t. to x' axis

I_{xC} : I w.r.t. to the centroid

$$\boldsymbol{I_{x'A} = I_{xCA} + A_A y_{CA}^2} = \frac{75(50)^3}{12} + 75(50)(105)^2 = 42.13(10^6) \text{ mm}^4$$

$$\boldsymbol{I_{x'B} = I_{xCB} + A_B y_{CB}^2} = \frac{25(150)^3}{12} + 25(150)(5)^2 = 7.13(10^6) \text{ mm}^4$$

$$\boldsymbol{I_{x'C} = I_{xCC} + A_C y_{CC}^2} = \frac{200(25)^3}{12} + 200(25)(-82.5)^2 = 34.29(10^6) \text{ mm}^4$$

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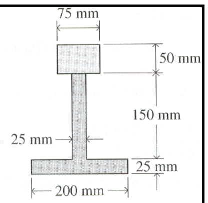
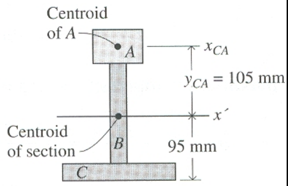
Example Problem 7-3 (IV)



- Compute $I_{x'}$

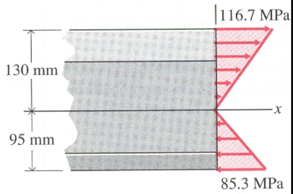
$$I_{x'} = I_{x'A} + I_{x'B} + I_{x'C} = 42.13(10^6) + 7.13(10^6) + 34.29(10^6)$$
$$= 83.55(10^6)\,\text{mm}^4 = 83.55(10^{-6})\,\text{m}^4$$



- Compute σ_{max} (Tensile) and σ_{max} (Compressive)

$$\sigma_{max}(\text{T}) = -\frac{M_r y_t}{I} = -\frac{-75(10^3)(130)(10^{-3})}{83.55(10^{-6})}$$

$$\cong 116.7 \text{ MPa (T)}$$



$$\sigma_{max}(\text{C}) = -\frac{M_r y_b}{I} = -\frac{-75(10^3)(-95)(10^{-3})}{83.55(10^{-6})}$$

$$\cong 85.3 \text{ MPa (C)}$$

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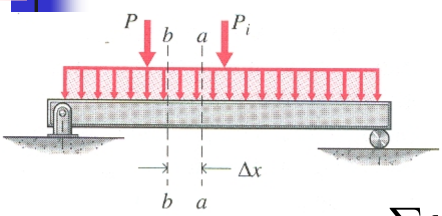
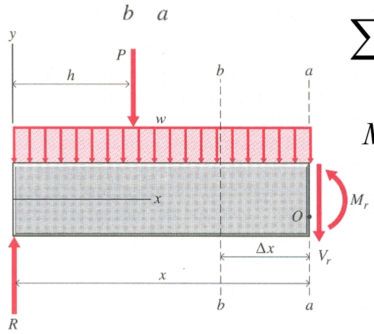
7-5 Shear Forces and Bending Moments in Beams



$$\sum F_y = R - wx - P - V_r = 0$$

$$V_r = R - wx - P = V$$



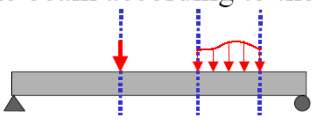
$$\sum M_O = -Rx + \frac{wx^2}{2} + P(x-h) + M_r = 0$$

$$M_r = Rx - \frac{wx^2}{2} - P(x-h) = M$$

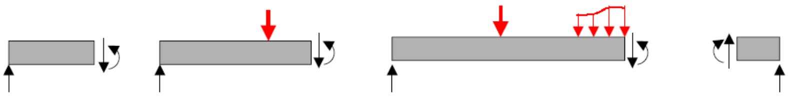
V_r, M_r: resisting force and moment
V, M: resultant force and moment at the given section

Procedure to determine M(x) and V(x)

- Divide the beam according to the loads.



- For each section, draw a FBD.



- Apply equilibrium equations to each section.

$$\sum F_y = 0 \qquad \Rightarrow \quad V(x)$$

$$\sum M_{\text{cut}} = 0 \qquad \Rightarrow \quad M(x)$$

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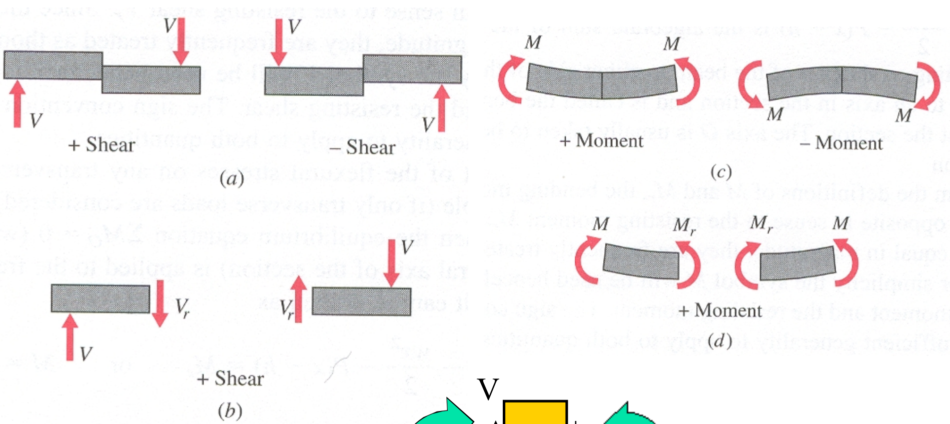
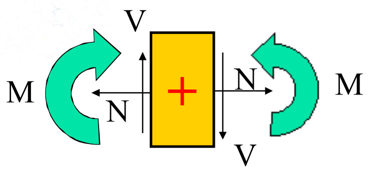
Sign Convention





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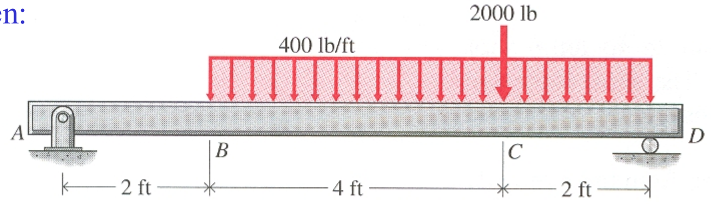
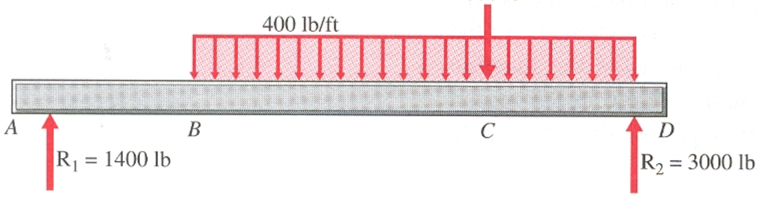
Example Problem 7-5 (I)

Given:



- $\Sigma M_D = 0$, $\Sigma M_A = 0$ ⇨ $R_1 = 1400$ lb and $R_2 = 3000$ lb



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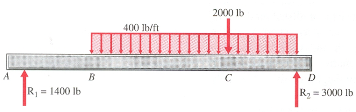
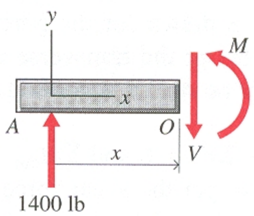
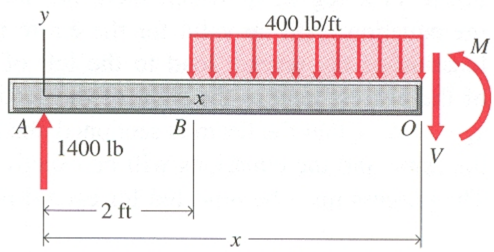
Example Problem 7-5 (II)







AB: $0 < x < 2$

$$\sum F_y = 1400 - V = 0$$

$$V = 1400 \text{ lb}$$

$$\sum M_O = -1400x + M = 0$$

$$M = 1400x \text{ ft} \cdot \text{lb}$$

BC: $2 < x < 6$

$$\sum F_y = 1400 - 400(x-2) - V = 0$$

$$V = -400x + 2200 \text{ lb}$$

$$\sum M_O = -1400x + 400(x-2)\left(\frac{x-2}{2}\right) + M = 0$$

$$M = -200x^2 + 2200x - 800 \text{ ft} \cdot \text{lb}$$

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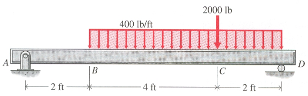
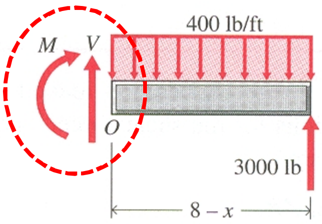
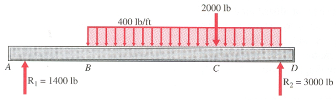
Example Problem 7-5 (III)



CD: $6 < x < 8$





$$\sum F_y = V - 400(8-x) + 3000 = 0 \qquad V = -400x + 200 \text{ lb}$$

$$\sum M_O = -M - 400(8-x)\left(\frac{8-x}{2}\right) + 3000(8-x) = 0$$

$$M = -200x^2 + 200x + 11{,}200 \text{ ft} \cdot \text{lb}$$

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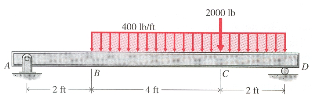
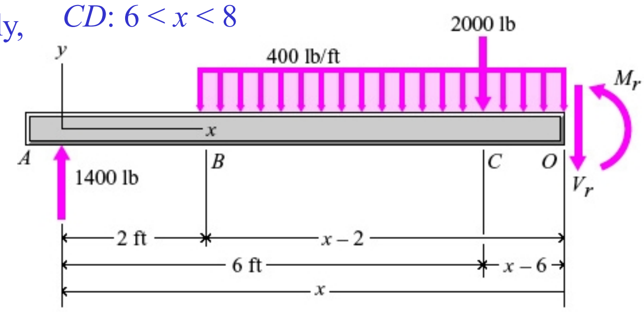
Example Problem 7-5 (IV)



Alternatively, *CD*: $6 < x < 8$



$$\sum F_y = 1400 - 400(x-2) - 2000 - V_r = 0 \qquad V_r = -400x + 200 \text{ lb}$$

$$\sum M_O = -1400x + 400(x-2)\left(\frac{x-2}{2}\right) + 2000(x-6) + M_r = 0$$

$$M_r = -200x^2 + 200x + 11{,}200 \text{ ft} \cdot \text{lb}$$

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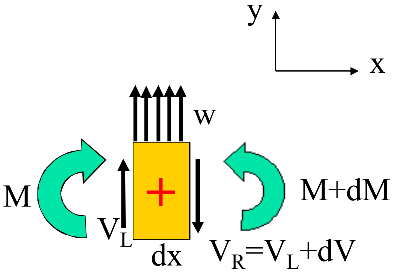
7-6 Shear Forces and Bending Moments in Beams

- If distributed load exists



$$\sum F_y = \cancel{V_L} + w dx - (\cancel{V_L} + dV) = 0$$

$$\boxed{\frac{dV}{dx} = w}$$

$$V_2 - V_1 = \int_{V_1}^{V_2} dV = \int_{x_1}^{x_2} w\, dx$$

$$\sum M_L = 0$$

$$-\cancel{M} + \cancel{wdx(\frac{dx}{2})} - (V_L + \cancel{dV})dx + (\cancel{M} + dM) = 0$$

$$\Rightarrow \boxed{\frac{dM}{dx} = V_L}$$

How about using $\sum M_R = 0$?

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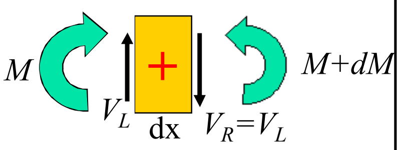
7-6 Shear Forces and Bending Moments in Beams

- No concentrated load or distributed load

$$\frac{dV}{dx} = 0 \quad \Delta V = V_R - V_L = 0$$

$$\frac{dM}{dx} = V_L = V_R \quad \text{or} \quad \boxed{\frac{dM}{dx} = V}$$

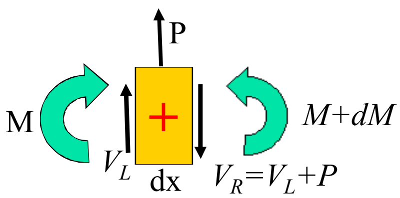


- If concentrated load exists

$$V_R = V_L + P$$

$$\Delta V = V_R - V_L = P$$

$$\left.\frac{dM}{dx}\right)_L = V_L \qquad \left.\frac{dM}{dx}\right)_R = V_R$$



$$M_2 - M_1 = \int_{M_1}^{M_2} dM = \int_{x_1}^{x_2} V\, dx$$

Change of Moment = area under V-diagram

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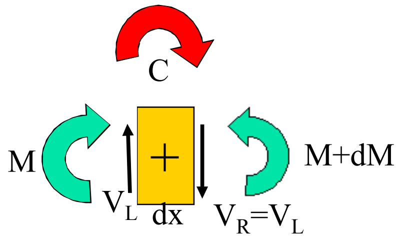
7-6 Shear Forces and Bending Moments in Beams

- If concentrated couple exists, $C \neq 0$

$$\Delta V = V_R - V_L = 0$$

$$\Delta M = C$$

$$M_R = M_L + C$$



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Shear and Bending Moment Diagrams

Dictate variations of shear and bending moment.

- Method 1 (代數法):
  - Establish algebraic equations for shear forces and bending moments.
  - Construct curves from the equations.
- Method 2 **(圖解法，Convenient)** :
  - Draw the free body diagram for the entire beam with applied loads and reactions on proper coordinates.
  - Construct the shear diagram directly below the FBD.
  - Plot the bending moment diagram further below.

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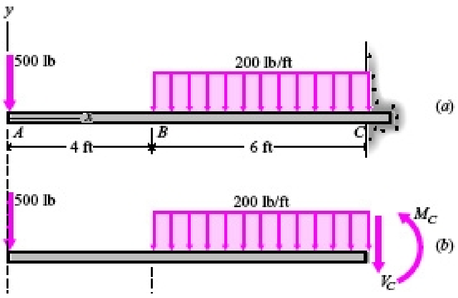
Example Problem 7-7

Given:



Find:

- Write $V(x)$ and $M(x)$ in the interval AB
- Write $V(x)$ and $M(x)$ in the interval BC
- Draw V and M diagrams

Sol: $\sum F_y = 0: \quad -500 - 200 \cdot 6 - V_C = 0$

$$V_C = -1700 \text{ lb}$$

$\sum M_C = 0: \quad 500 \cdot 10 + 200 \cdot 6 \cdot 3 + M_C = 0$

$$M_C = -8600 \text{ lb} \cdot \text{ft}$$

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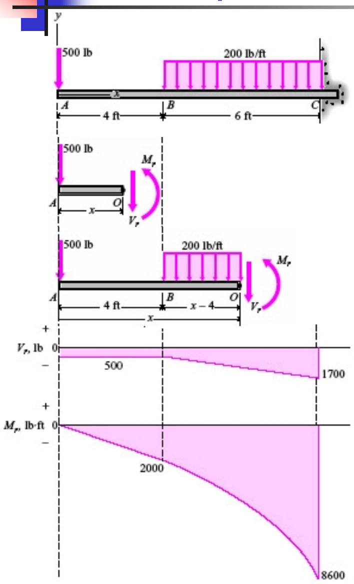
Example Problem 7-7

$V_C = -1700$ lb

$M_C = -8600$ lb · ft



AB: $0 < x < 4$ ft

$\sum F_y = 0: \quad -500 - V_r = 0$

$$V_r = -500 \text{ lb}$$

$\sum M_O = 0: \quad 500 \cdot x + M_r = 0$

$$M_r = -500 \cdot x \text{ lb} \cdot \text{ft}$$

BC: $4 < x < 10$ ft

$\sum F_y = 0: \quad -500 - 200(x-4) - V_r = 0$

$$V_r = 300 - 200x \text{ lb}$$

$\sum M_O = 0: \quad 500 \cdot x + 200(x-4) \cdot (x-4)/2 + M_r = 0$

$$M_r = -100x^2 + 300 \cdot x - 1600 \text{ lb} \cdot \text{ft}$$

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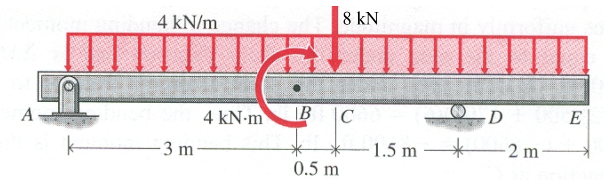
Example Problem 7-8 (I)

Given:



Find:
- Write equations of $V(x)$ and $M(x)$ in the interval CD
- Draw V and M diagrams

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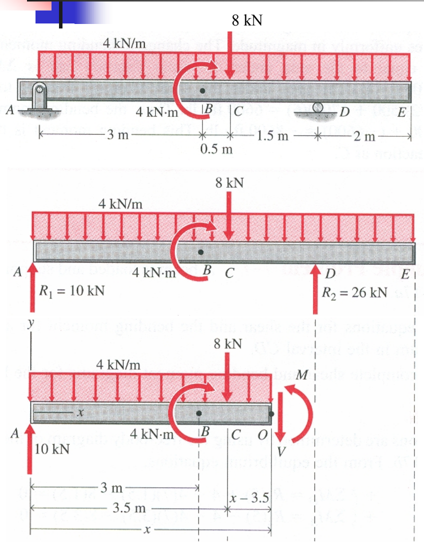
Example Problem 7-8 (II)



(+)

$$\sum \boldsymbol{M_D} = -5\boldsymbol{R}_1 - 4 + 4(7)(1.5) + 8(1.5) = 0$$

$$\sum \boldsymbol{M_A} = 5\boldsymbol{R}_2 - 4 - 4(7)(3.5) - 8(3.5) = 0$$

$$R_1 = 10\text{ kN} \qquad R_2 = 26\text{ kN}$$

CD: $3.5 < x < 5$ m

$$\sum \boldsymbol{F_y} = 10 - 4\boldsymbol{x} - 8 - \boldsymbol{V} = 0$$

$$\boldsymbol{V} = -\,4\boldsymbol{x} + 2\text{ kN}$$

(+) $$\sum M_O = -10(x) + 4x(x/2) - 4 + 8(x - 3.5) + M = 0$$

$$M = -2x^2 + 2x + 32\text{ kN}\cdot\text{m}$$

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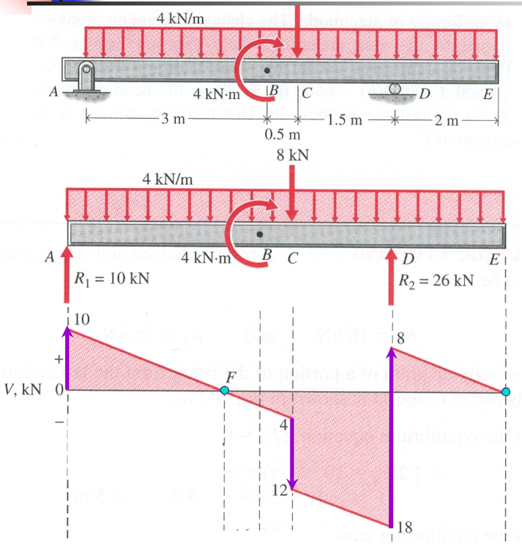
Example Problem 7-8 (III)



Draw V diagram:

$$V_C^- = 10 - 4(3.5) = -4 \text{ kN}$$

$$V_C^+ = -4 - 8 = -12 \text{ kN}$$

$$V_D^- = -12 - 4(1.5) = -18 \text{ kN}$$

$$V_D^+ = -18 + 26 = 8 \text{ kN}$$

$$V_E = +8 - 4(2) = 0 \text{ kN}$$

$$x_1 = 10/4 = 2.5 \text{ m}$$

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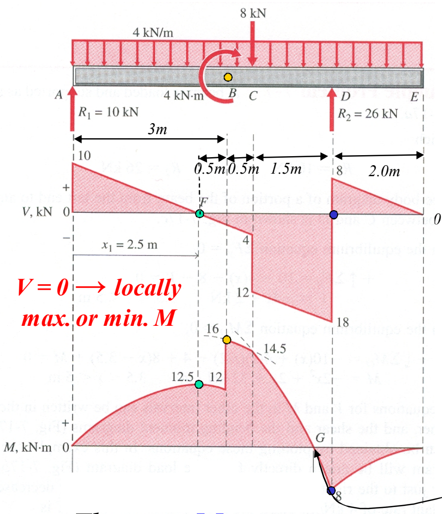
Example Problem 7-8 (IV)

Draw M diagrams ***Change of Moment = area under V-diagram***



V = 0 ⟶ locally max. or min. M

$$M_F = 0 + 10(2.5)/2 = 12.5 \text{ kN}\cdot\text{m}$$

$$V_B = 10 - 4(3) = -2 \text{ kN}$$

$$M_{B\_} = 12.5 - 2(0.5)/2 = 12.0 \text{ kN}\cdot\text{m}$$

$$M_{B+} = 12.0 + 4 = 16.0 \text{ kN}\cdot\text{m}$$

$$M_C = 16.0 - (4+2)(0.5)/2 = 14.5 \text{ kN}\cdot\text{m}$$

$$M_D = 14.5 - (12+18)(1.5)/2 = -8.0 \text{ kN}\cdot\text{m}$$

$$M_E = -8.0 + 8(2)/2 = 0 \text{ kN}\cdot\text{m}$$

Solving $M = -2x^2 + 2x + 32 \; kN\cdot m = 0$

$$x_2 = 0.5 \pm \sqrt{16.25} \cong 4.531 \text{ m}$$

M = 0, for sections to be spliced

*The **max M** occurs at position B where **V ≠0** in this problem!*

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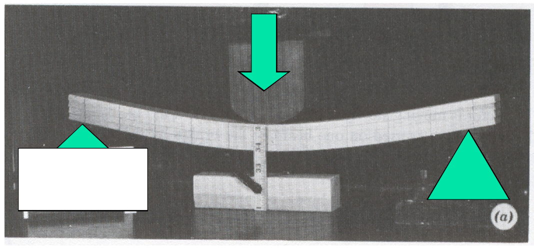
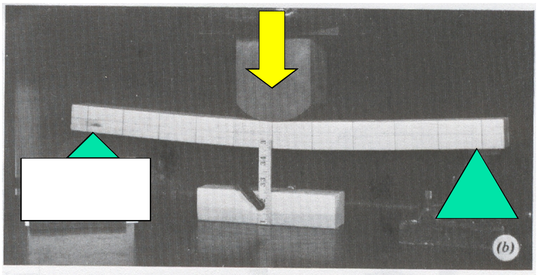
Flexural Stresses: σ_x

7-7 Shearing Stresses in Beams



- Stacking flat slabs (平板) **without gluing**.
- Load: 200 lb
- Deflection: 1.18"
- Note rugged ends

laminated beam



- Stacking flat slabs **with gluing**.
- Load: 800 lb
- Deflection: 0.375"

Why? Glue-laminated beam

shearing between slabs?

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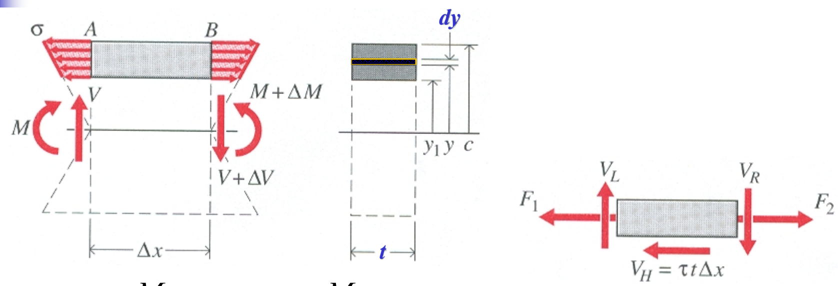
7-7 Shearing Stresses in Beams (I)



$$F_1 = -\frac{M}{I}\int_A y\,dA = -\frac{M}{I}\int_{y_1}^{c} y\,(t\,dy)$$

$$F_2 = -\frac{M+\Delta M}{I}\int_{y_1}^{c} y\,(t\,dy)$$

$$F_x = \int_A \sigma_x dA$$

$$\sigma_x = -\frac{My}{I}$$

Force balance: $$V_H = F_2 - F_1 = -\frac{\Delta M}{I}\int_{y_1}^{c} y\,(t\,dy)$$

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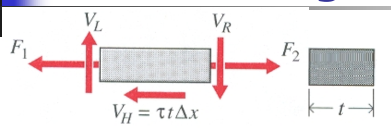
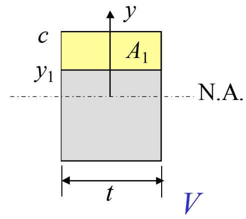
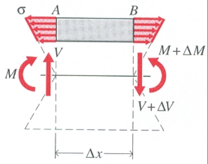
7-7 Shearing Stresses in Beams (II)







$$V_H = -\frac{\Delta M}{I}\int_{y_1}^{c} y\,(t\,dy) \qquad A_s = t\Delta x$$

$$\tau_{avg} = \frac{V_H}{A_s} = -\frac{\Delta M}{It\Delta x}\int_{y_1}^{c} y\,(t\,dy)$$

$$\tau_{(H)} = \lim_{\Delta x\to 0}\frac{V_H}{A_s} = \lim_{\Delta x\to 0}\frac{\Delta M}{\Delta x}\left(-\frac{1}{It}\right)\int_{y_1}^{c} ty\,dy = \frac{dM}{dx}\left(-\frac{1}{It}\right)\int_{y_1}^{c} ty\,dy$$

$\frac{dM}{dx}$: V

Q : 1st moment of A_1

$$\tau_{(H)} = -\frac{VQ}{It}$$

$$\tau = \frac{VQ}{It}$$

Let sense of τ = sense of V

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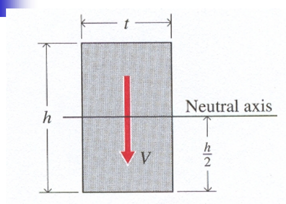
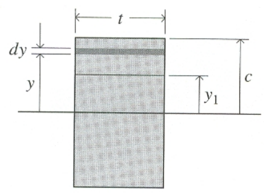
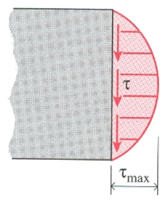
Shearing Stress in Rectangular Beam







$$\tau = \frac{VQ}{It} = \frac{V}{It}\int_{y_1}^{c} ty\,dy = \frac{V}{I}\int_{y_1}^{h/2} y\,dy = \frac{V}{2I}\left[\left(\frac{h}{2}\right)^2 - y_1^2\right]$$

at $y_1 = 0$ $\quad \tau_{max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8(th^3/12)} = \frac{3}{2}\frac{V}{th} = \frac{3}{2}\frac{V}{A}$

$h = 2t$, error 3%, $h = t$, error 12%, $h = t/4$, **error 100%**

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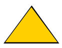
Remarks

- $\tau = \dfrac{V_r Q}{It}$ is only valid for elastic action.
- τ is the **average** shearing stress across the thickness t and is ***only accurate if t is not too big (wide)***.
- $\tau = \dfrac{V_r Q}{It}$ should ***not*** be applied to a **wide** cross-section, flange of I or T beam, or section where the sides are not parallel, e.g., 
- Although the shear stress equation is derived for the horizontal shear area, it is also valid for the vertical transverse section of a beam. $\because \tau_{xy} = \tau_{yx}$

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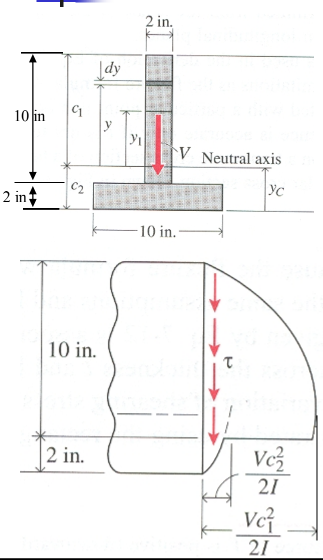
Shearing Stress in T-shaped Beam



$$y_c = \frac{2(10)(7) + 10(2)(1)}{2(10) + 10(2)} = 4 \text{ in.}$$

$c_1 = 8$ in. $\qquad c_2 = 4$ in.

$$\tau = \frac{VQ}{It} = \frac{V}{It}\int_{y_1}^{c} ty\,dy$$

$$= \frac{V}{2I}\left(c_1^2 - y_1^2\right) = \frac{V}{2I}\left(8^2 - y_1^2\right) \quad \text{stem}$$

$$\tau = \frac{V}{2I}\left(c_2^2 - y_1^2\right) = \frac{V}{2I}\left(4^2 - y_1^2\right) \quad \text{flange}$$

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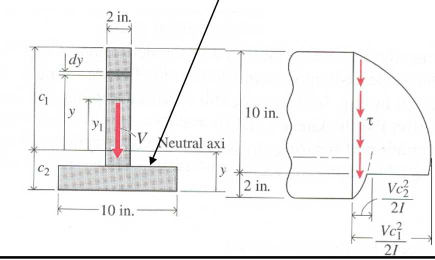
Shearing Stress in T-shaped Beam

- Stress distribution: parabolic
- Stress discontinuity at the junction of flange and stem.
- Distribution in the flange is fictitious. Flange top should be stress free.
- In general, τ_{max} occurs at the neutral surface.

  exception: X-shaped beam.



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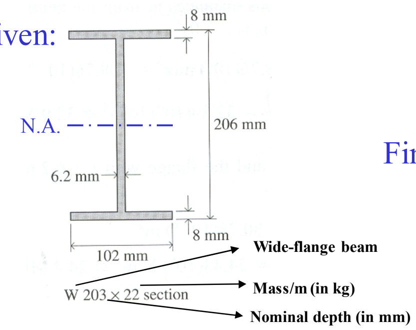
Shearing Stress in I-Beam

Given:



$V = 37.5$ kN

$I = 20.0(10^6)$ mm^4

Find:

- τ at various points

Please do it by yourself

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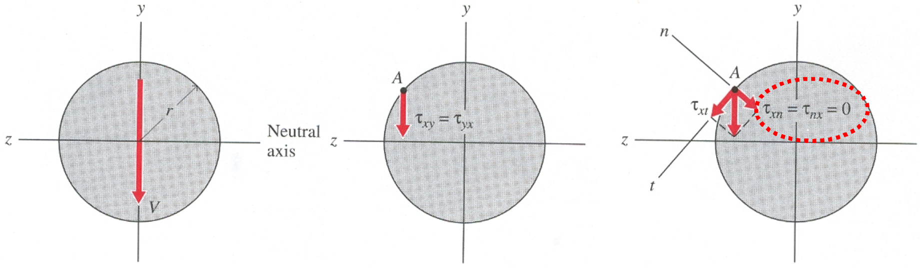
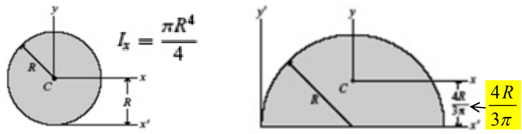
Shearing Stress in Circular Beam



$$\tau_{max} = \tau_{NA} = \frac{VQ_{NA}}{It_w} = \frac{V(\pi r^2/2)(4r/3\pi)}{(\pi r^4/4)(2r)} = \frac{4V}{3\pi r^2} = \frac{4}{3}\frac{V}{A}$$

$$\tau_{max}(\text{exact}) = 1.38\frac{V}{A}$$

(Table A-1, p. A12; Table A-2, p. A19)



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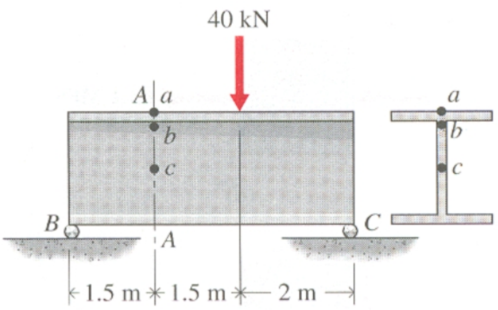
Example Problem 7-9 (I)



- W254×33 beam

At section *A*–*A*,

Determine σ and τ at

- Point *a* on the top of the flange
- Point *b* in the web at the junction
- Point *c* on the N.A.

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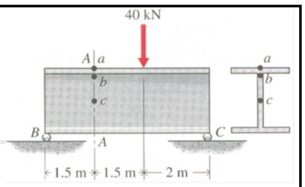
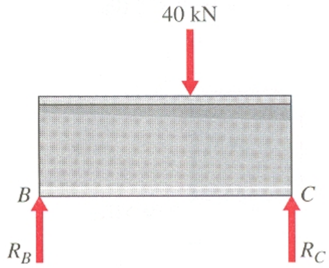
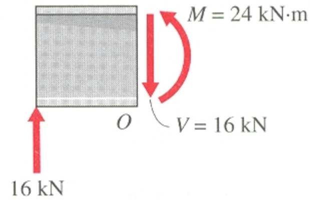
Example Problem 7-9 (II)





$$\sum M_C = 40(2) - R_B(5) = 0$$
$$\sum M_B = R_C(5) - 40(3) = 0$$

$$R_B = 16\text{ kN} \qquad R_C = 24\text{ kN}$$



$$\sum F_y = 16 - V = 0$$
$$\sum M_O = M - 16(1.5) = 0$$

$$V = 16\text{ kN} \qquad M = 24\text{ kN}\cdot\text{m}$$

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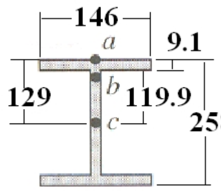
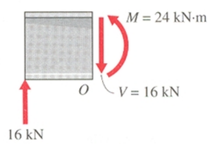
Example Problem 7-9 (III)

W254×33 section: (p. A27):

$I = 49.1(10^6)$ mm^4, d (depth) = 258 mm,
w_f = 146mm, t_f = 9.1 mm, t_w = 6.1 mm



- **Point *a*** (On the top surface)

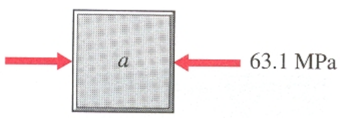


$y = d/2 = 129$ mm

$$\sigma_a = -\frac{My}{I} = -\frac{24(10^3)(129)(10^{-3})}{49.1(10^{-6})}\text{ N/m}^2 \cong 63.1\text{ MPa (C)}$$

$Q_a = 0$

$\tau = 0$



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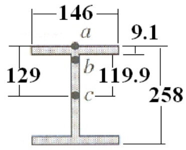
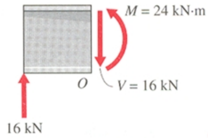
Example Problem 7-9 (IV)



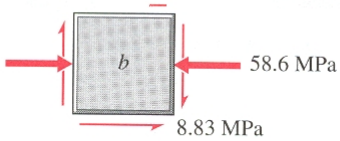


- **Point *b*** (at junction, in the web)

  $I = 49.1(10^6)\ \text{mm}^4,\ d = 258\ \text{mm}$

  $w_f = 146\text{mm},\ t_f = 9.1\ \text{mm},\ t_w = 6.1\ \text{mm}$

  $y = 129 - 9.1 = 119.9\ \text{mm}$



$$\sigma_b = -\frac{My}{I} = -\frac{24(10^3)(119.9)(10^{-3})}{49.1(10^{-6})}\ \text{N/m}^2 \cong 58.6\ \text{MPa (C)}$$

$$Q_b = y_C A = (119.9 + 9.1/2)(146)(9.1)\ \text{mm}^3 = 165.34(10^{-6})\ \text{m}^3$$

$$\tau_b = \frac{VQ_b}{I_{NA}t_w} = \frac{16(10^3)(165.34)(10^{-6})}{49.1(10^{-6})(6.1)(10^{-3})}\ \text{N/m}^2 \cong 8.83\ \text{MPa}$$

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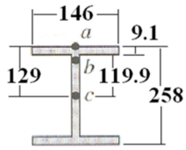
Example Problem 7-9 (V)

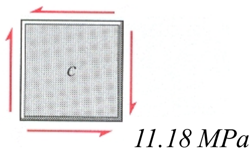


- Point *c* (on N.A.)

  $I = 49.1(10^6)\ \text{mm}^4,\ d = 258\ \text{mm}$

  $w_f = 146\text{mm},\ t_f = 9.1\ \text{mm},\ t_w = 6.1\ \text{mm}$

  $y = 0 \qquad \sigma_c = 0$



$$Q_c = y_c A = 124.45(146)(9.1) + 59.95(119.9)(6.1)\ \text{mm}^3 = 209.2(10^{-6})\ \text{m}^3$$

$$\tau_c = \frac{VQ_c}{I_{NA}t_w} = \frac{16(10^3)(209.2)(10^{-6})}{49.1(10^{-6})(6.1)(10^{-3})}\ \text{N/m}^2 \cong 11.18\ \text{MPa}$$

$$\tau_{ave} \cong \frac{V}{A_{web}} = \frac{16(10^3)}{6.1(10^{-3})(239.8)(10^{-3})}\ N/m^2 \cong 10.94\ MPa \quad \text{(approx.)}$$

(neglect flange area)

$$D = (11.176 - 10.938)/11.176(100) = 2.13\%$$

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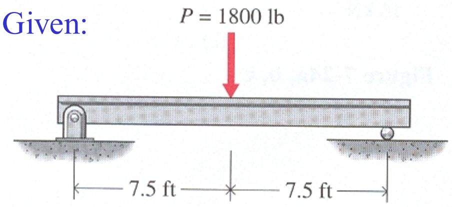
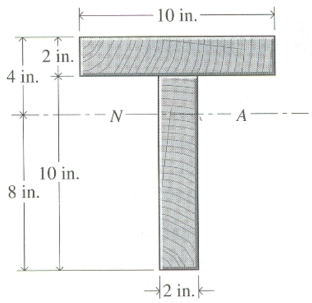
Example Problem 7-10 (I)

Given:





Determine

- τ_{avg} on a plane 4 in. above bottom and at $x = 6$ ft
- τ_{max} in the beam
- τ_{avg} in the joint between flange and stem and at $x = 6$ ft
- Force between flange and stem by the glue in a 12-in. length centered at $x = 6$ ft
- σ_{max} in the beam

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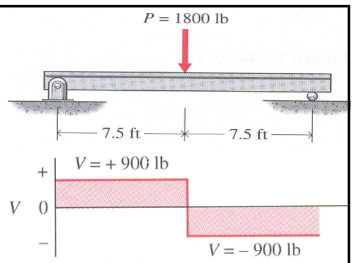
Example Problem 7-10 (II)



$$V(x = 6\text{ ft}) = 900\text{ lb}$$

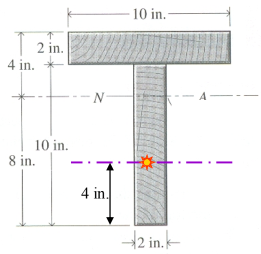
$$I_{NA} = \frac{1}{12}(2)(10)^3 + 2(10)(3)^2 + \frac{1}{12}(10)(2)^3 + 10(2)(3)^2 = 533.3\text{ in.}^4$$

- τ_{avg} on a plane 4 in. above bottom and at $x = 6$ ft

$$Q_4 = y_{C4}A_4 = 6(2)(4) = 48\text{ in.}^3$$

$$\tau_4 = \frac{VQ_4}{I_{NA}t_4} = \frac{900(48)}{533.3(2)} \cong 40.5\text{ psi}$$

(sense of τ = sense of V)



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Example Problem 7-10 (III)

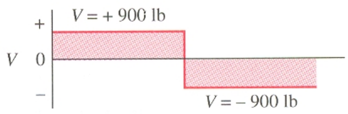
- τ_{max} in the beam (occurs at N.A.)

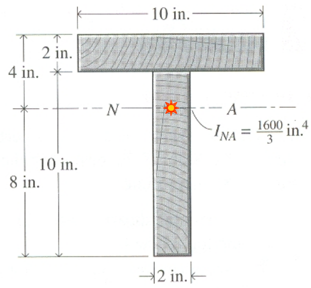
$$V_{max} = 900 \text{ lb}$$

$$Q_{NA} = 4(2)(8) = 64 \text{ in.}^3 \quad \text{(lower)}$$

or $$Q_{NA} = 3(10)(2) + 1(2)(2) = 64 \text{ in.}^3 \quad \text{(upper)}$$

$$\tau_{max} = \frac{VQ_{NA}}{I_{NA}t_s} = \frac{900(64)}{533.3(2)} \cong 54.0 \text{ psi}$$





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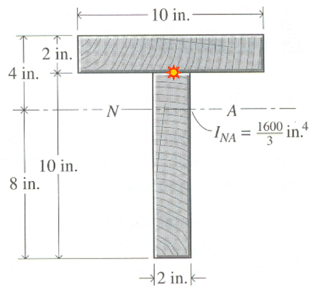
Example Problem 7-10 (IV)

- τ_{avg} in the joint between flange and stem and at $x = 6$ ft

$$Q_F = y_{CF}A_F = 3(10)(2) = 60 \text{ in.}^3$$

$$\tau_J = \frac{VQ_F}{I_{NA}t_s} = \frac{900(60)}{533.3(2)} \cong 50.6 \text{ psi}$$



- Force between flange and stem by the glue (12 in-length)

$$V_g = \tau_J A_J = 50.63(12)(2) \cong 1215 \text{ lb}$$

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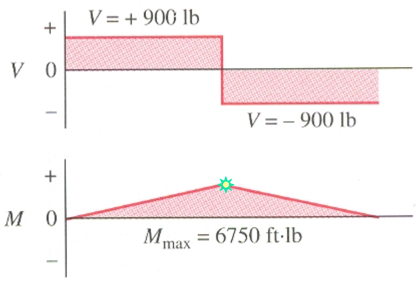
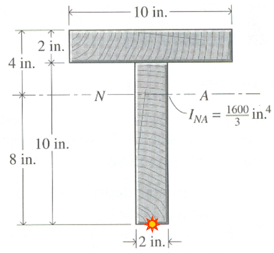
Example Problem 7-10 (V)

- σ_{max} in the beam (M_{max}, c_{max})





$$\sigma_{max} = \frac{M_{max}c}{I} = \frac{6750(12)(8)}{533.3} = 1215.1\ \text{psi} \cong 1215\ \text{psi (T)}$$

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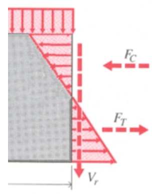
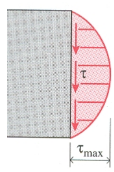
7-8 Principal Stresses in Flexural Members (I)

- On the top or bottom edge of a section
  - $\rightarrow \sigma$ is maximum, $\tau = 0$
  - $\rightarrow \sigma_p = \sigma$, $\tau_p = (\sigma_p - 0)/2$
- At the neutral axis
  - $\rightarrow \sigma = 0$, τ is maximum
  - $\rightarrow \sigma_p = \tau_p = \tau$
- What about other points?





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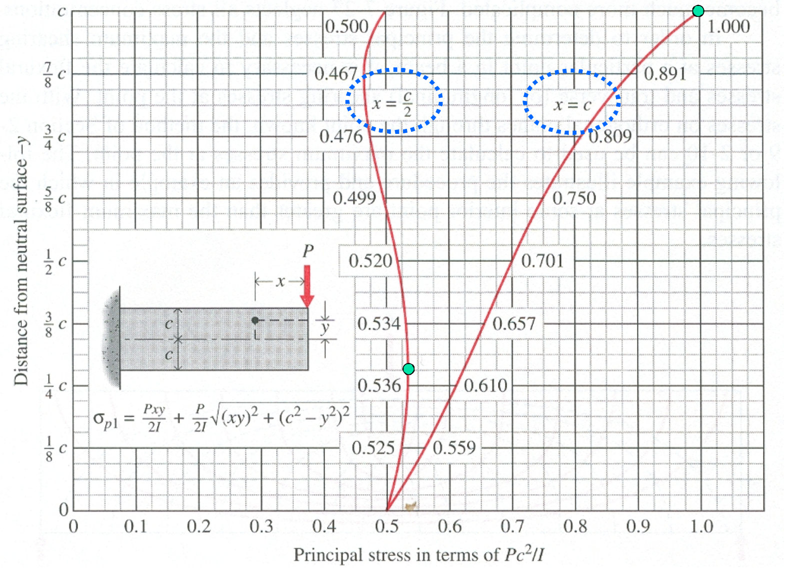
7-8 Principal Stresses in Flexural Members (II)

Departure from the flexural formula due to **Saint-Venant's principle**



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7-8 Principal Stresses in Flexural Members (III)

- For a section too close to P, the flexure formula doesn't apply. (Saint Venant's principle) $\sigma_x(y) = -\frac{M_r y}{I}$
- For a rectangular cross section, in regions where the flexure formula applies,

  max. normal stress = max. flexural stress

  τ_{max} = ½ (max. normal stress) usually.

  Sometimes, $\tau_{longitudinal}$ is the significant stress (e.g. timber)
- For deep, wide-flanged section with **large V and M**, σ_p may occur at the junction, ***not on the surface***.

  (large M, V, Q, and small t) $\tau = \frac{VQ}{It}$

  •Even though $\sigma < \sigma_{surface}$, τ is much larger than $\tau_{surface}$

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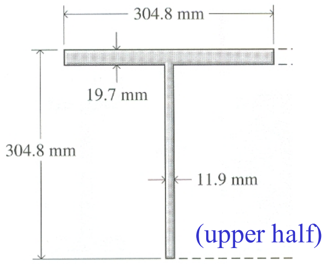
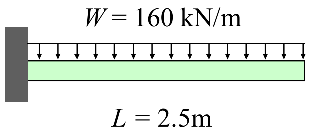
Example Problem 7-11 (I)





Given:

- W610×145 cantilever beam
  - $I = 1243(10^6)\ mm^4$
  - $S = 4079\ (10^3)\ mm^3$

Find:

- Max. normal and shearing stresses = ?

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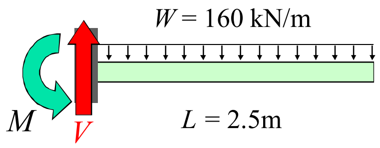
Example Problem 7-11 (II)



- Maximum V and M occur on the section at the support

$$M = -\frac{wL^2}{2} = -\frac{160(2.5^2)}{2} = -500\ \text{kN}\cdot\text{m}$$

$$V = wL = 160(2.5) = 400\ \text{kN}$$

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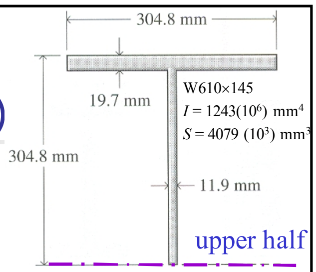
Example Problem 7-11 (III)



- At the neutral axis

$\sigma = 0$

$$Q_{NA} = 304.8(19.7)(295) + 285.1(11.9)(142.6) = 2.255(10^6)\,\text{mm}^3$$

$$\tau = \frac{VQ}{It} = \frac{400(10^3)(2.255)(10^{-3})}{1243(10^{-6})(11.9)(10^{-3})}\,\text{N/m}^2 \cong 61.0\ \text{MPa}$$

- At the top surface

$\tau = 0$

$$\sigma = -\frac{M}{S} = -\frac{-500(10^3)}{4079(10^{-6})}\,N/m^2 \cong 122.6\ MPa\ (T)$$

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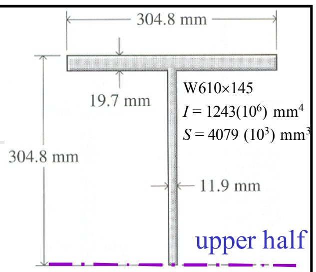
Example Problem 7-11 (IV)



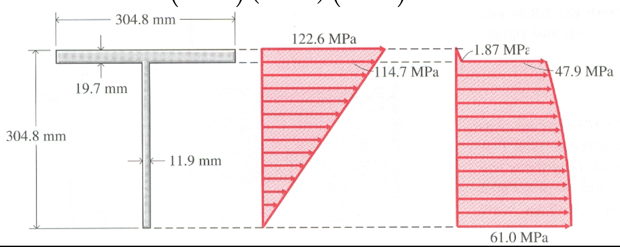
- In the web at the junction

$y = 304.8 - 19.7 = 285.1\text{mm}$

$$\sigma = -\frac{My}{I} = -\frac{-500(10^3)(0.2851)}{1243(10^{-6})}\,N/m^2 = 114.68 \cong 114.7\ MPa\ (T)$$

$$Q_J = 304.8(19.7)(295) = 1.771(10^6)\,\text{mm}^3$$

$$\tau = \frac{VQ_J}{It} = \frac{400(10^3)(1.771)(10^{-3})}{1243(10^{-6})(11.9)(10^{-3})}\,\text{N/m}^2 = 47.89 \cong 47.9\ \text{MPa}$$



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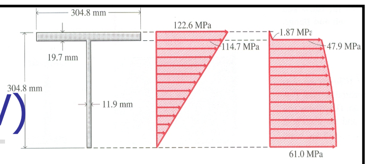
Example Problem 7-11 (V)



- At the junction σ =114.68 MPa, τ = 47.89 MPa

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{114.68+0}{2} \pm \sqrt{\left(\frac{114.68-0}{2}\right)^2 + (-47.89)^2}$$

$$= 132.05\ MPa\ (T),\ 17.37\ MPa\ (C)$$ σ =132.1 MPa

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{132.05-(-17.37)}{2} \cong 74.7\ \text{MPa}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{1}{2}\tan^{-1}\frac{2(-47.89)}{114.68-0} = -19.93°$$

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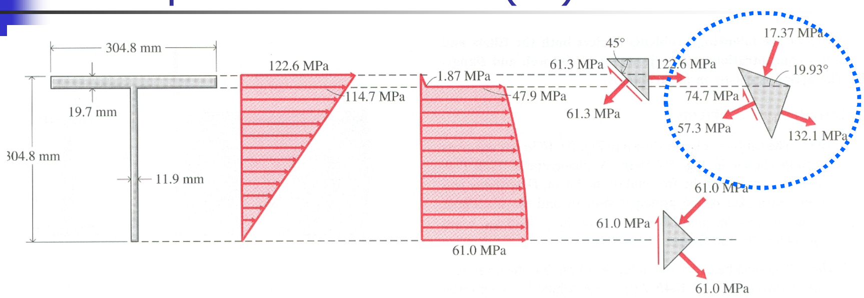
Example Problem 7-11 (VI)



Stress	Top Edge	Junction	Neutral Axis
σ_{p1}	122.6 MPa (T)	132.1 MPa (T)	61.0 MPa (T)
σ_{p2}	0	17.37 MPa (C)	61.0 MPa (C)
τ_{max}	61.3 MPa	74.7 MPa	61.0 MPa

8 Exercises

- **7-22, 7- 39, 7-41, 7-58, 7- 66, 7- 85, 7-89, 7-197**

"Warp"

Hibbeler, Mechanics of Materials (1997)